PHYSICS 715

Problem Set 10 **Due Friday, April 21, 2006**

Reading: Landau and Lifshitz, Secs. 57-61, 63, 106-109

LD 32: Characteristic temperatures for some fermi systems

(a) Calculate the Fermi energies E_F and the Fermi temperatures T_F for:

(i) liquid ^{3}He ($\rho = 0.0823$ gm/cm^3);

(ii) electrons in tungsten (valence 2, density 19.3 gm/cm^3);

(iii) neutrons in a large nucleus ($N \approx 0.6A$, radius $R = 1.2A^{1/3} \times 10^{-13}$ cm);

The work function of tungsten is 4.5 V. The binding energy of the last neutron in a large nucleus is ≈ 8 MeV. Determine the depths V_0 of the average or square-well potentials seen by:

(iv) the electrons in tungsten;

(v) and the neutrons in the nucleus.

Include a labelled sketch (energy-level diagram) to explain your reasoning.

(b) Long-chain molecules with some mobile electrons give one-dimensional organic conductors. Derive expressions for p_{F} and E_{F} for a one-dimensional conductor with n free electrons per unit length. Evaluate E_{F} for an atomic spacing of 2.5 Å between donor atoms and 0.5 free electron per atom.

LD 33: ^{3}He in ^{4}He: the Fermi gas model for a quantum liquid.

Pure liquid ^{3}He and ^{3}He dissolved in liquid ^{4}He at low temperatures are well described as ideal Fermi liquids with single-particle energies

$$E(p) = \frac{p^2}{2m_3^*} - \epsilon \ ,$$

where the effective mass m_3^* of an atom of ^{3}He and the constant ϵ both depend on the liquid in question. (The model was introduced by Landau.)

(a) Estimate the number density $n(^3\mathrm{He})$ and the fraction $n(^3\mathrm{He})/n(^4\mathrm{He})$ of ^{3}He atoms (nuclear spin 1/2) which can be dissolved in liquid ^{4}He at ~ 0 K by considering the equilibrium of the dissolved ^{3}He with liquid ^{3}He. The binding energy of a single ^{3}He atom in liquid ^{4}He is 2.79 K (temperature units). The binding energy of a ^{3}He atom in liquid ^{3}He (that is, the minimum energy necessary to remove an atom from the liquid) is 2.47 K. $m_3^*/m_3 = 2.34$ for ^{3}He dissolved in ^{4}He. The density of liquid ^{4}He is $\rho_4 = 0.145$ gm/cm^3. [Hint: sketch the energy diagrams for the two systems, and relate the binding energies to the chemical potentials at 0 K.]

(b) The heat capacity per atom of liquid ^{3}He is $c = (3.05\ \mathrm{K}^{-1})kT$ for $T \sim 0$. $\rho_3 = 0.0823\ \mathrm{gm/cm}^2$. Calculate the effective mass m_3^* for ^{3}He atoms in the liquid in units of m_3.

(c) Calculate the vapor pressure in Torr (1 atm = 760 Torr) of gaseous ^{3}He in equilibrium with liquid ^{3}He for $0 \leq T \leq 0.6$ K. [Hint: Recall that $\mu \neq -\epsilon + E_F$ for $T > 0$. Include the corrections to μ to order T^2.] Plot $\log P$ vs. T. Some data for comparison are $P = 1.2 \times 10^{-5}$, 2.81×10^{-2}, 0.544 Torr at $T = 0.2, 0.4, 0.6$ K.

For more information on the quantum liquids see *The Physics of Liquid and Solid Helium*, K.H. Bennemann and J.B. Ketterson, editors, especially the article by G. Baym and C. Pethick, Vol. II, p. 123.

The 1996 Nobel Prize in Physics was awarded to David Lee, Douglas Osheroff, and Robert Richardson for the discovery of the superfluid phase transition in liquid ^{3}He ("Last month's Nobel Prize is this month's homework problem" – well, almost – LD), an analog of the superconducting phase transition for electrons in a superconductor. See the article by Lee and Richardson in the reference above, Vol. II, p. 287.

LD 34: Pauli paramagnetism: spin paramagnetism in metals.

The spin magnetic moments of electrons in a metal lead to a weak paramagnetism (Pauli paramagnetism – W. Pauli, Z. Physik **41**, 81 (1927)). Consider an ideal electron gas described by a Hamiltonian with eigenvalues

$$E_\pm(p) = \frac{p^2}{2m} \pm \mu_B B,\ \ \mu_B = \frac{|e|\hbar}{2mc} = -\mu_e$$

for electron spin projections $\pm\frac{1}{2}$ along the direction of the magnetic field B.

(a) Use the grand canonical approach to calculate the chemical potential μ for the nearly degenerate electron gas to second order in kT and $\mu_B B$. Treat the electron gases with spins up and down as independent, with $N = N^+ + N^-$ fixed. Assume the field is weak, $\mu_B B \ll E_F$, and that $kT \ll E_F$, and expand appropriately. Express your result in terms of n and E_F.

(b) Derive the leading terms in the expressions for the magnetization and magnetic susceptibility of the degenerate electron gas, $M_{\mathrm{Pauli}} = -\mu_B(N^+ - N^-)/V$, $\chi_{\mathrm{Pauli}} = \partial M_{\mathrm{Pauli}}/\partial B$. Express the results as multiples of $\mu_B n$, the total magnetic moment per unit volume. Estimate χ_{Pauli} for copper at 300 K (valence 1, density 8.94 $\mathrm{gm/cm}^3$, atomic weight 63.5).

LD 35: Stability of a neutron star.

A neutron star can be treated approximately as a sphere of non-relativistic neutrons which is bound gravitationally, and is prevented from collapsing by the neutron Fermi pressure.

(a) Assume that the neutron mass density $\rho(r)$ decreases linearly with the radial coordinate r, $\rho(r) = \rho(0)\left(1 - \frac{r}{R}\right)$, where R is the radius of the star, and estimate R for a neutron star of mass $M = 1.4\, M_\odot$ at $T = 0$ by calculating the gravitational pressure at the center, and requiring that it be balanced by the Fermi pressure of the neutrons. [Hint: Integrate the equation $dP/dr = -G\rho(r)M(r)/r^2$ inward from $r = R$ to determine $P(r)$. Here $M(r)$ is the total mass inside the radius r.] Determine the range of radii r for which the neutrons are "cold" from point of view of statistical mechanics even at a typical temperature of 10^7 K.

$$\begin{aligned} M_\odot &= \text{mass of sun} = 2 \times 10^{33}\text{g} \\ G &= 6.67 \times 10^{-8}\text{dynes}\,\text{cm}^2/\text{g}^2, \qquad m_n = 1.68 \times 10^{-24}\text{g}. \end{aligned}$$

(b) Neutron stars contain a small admixture of electrons and protons in equal numbers. The following reactions are possible in principle,

$$\begin{aligned} &n \to p + e^- + \bar{\nu}_e, \\ &p + e^- \to n + \nu_e. \end{aligned}$$

The neutrinos and antineutrinos emitted in these reactions would escape from the star if the reactions actually occurred, and the star would contract. Determine the equilibrium proton-to-neutron ratio $x = n_p/n_n$ at $T = 0$ at the center of the star of part (a) by using the conditions for "chemical" equilibrium. Treat the electrons as an ultrarelativistic degenerate Fermi gas with single-particle energies $E(p) = pc$, and find their chemical potential μ_e. $\mu_i = m_i c^2 + E_{F,i}$ for the nonrelativistic particles. Note that the neutrinos are not confined in the star. You may assume that $n_p/n_n \ll 1$. Explain why the reactions above do not occur at equilibrium even though the energy of the star would apparently be lowered by the energy carried off by the $\bar{\nu}$ or ν. [Hint: Consider the constraints from energy-momentum conservation.]

$$m_n = 939.566 \text{ MeV}, \qquad m_p = 938.272 \text{ MeV}, \qquad m_e = 0.511 \text{ MeV}$$

$$m_{\bar{\nu}} = m_\nu = 0.$$