LD 39: Bose-Einstein condensation in an atomic trap.

Bose-Einstein condensation was observed directly in 1995 in dilute atomic gases confined in magnetic traps (see M. H. Anderson et al., Science 269, 198 (1995); C. C. Bradley et al., Phys. Rev. Lett. 75, 1687 (1995); K. B. Davis et al., Phys. Rev. Lett. 75, 3969 (1995)). The traps are approximately harmonic with characteristic oscillation frequencies $\nu$ of about 150 Hz. The critical temperature $T_c$ at which the condensation starts varies with the experimental conditions, but is typically $\approx 150$ nK.

(a) A mean field model for an ideal Bose gas in an external potential $V(r)$ describes the gas in a volume $d^3r$ at a point $r$ as being in local equilibrium and having a local chemical potential $\mu(r) = \mu - V(r)$, where $\mu$ is the constant chemical potential for the entire system. When is this a reasonable approximation? The number distribution of the particles in the volume $d^3r$ is given in this picture by a standard Bose number distribution with $\mu$ replaced by $\mu(r)$, and the usual volume factor replaced by $d^3r$:

$$dN = \frac{1}{e^{E(p)} - \mu(r)} \frac{d^3p}{h^2} = \frac{1}{e^{H(p,r)} - \mu - 1} \frac{d^3p}{h^2}.$$ 

Integrate over all positions and momenta to obtain an expression for the maximum number of particles $N$ that can be accommodated in a harmonic trap with $V = \frac{1}{2}m\omega^2r^2$, and use the result to determine the critical temperature for Bose-Einstein condensation in terms of $N$ and the parameters in $H$. Evaluate $T_c$ for $N = 4 \times 10^4$ $^{87}$Rb atoms in a 150 Hz trap ($5S_{1/2}$ electron and a nucleus with spin $s = 3/2$ in a total angular momentum state with $F = 2, m_F = 2$). Compare the result to that in the first reference above. [Hint: the integral can be reduced to a standard form by introducing scaled variables $r' = r/\lambda, p' = \lambda p, \lambda = 1/\sqrt{m\omega}$, introducing the the 6-dimensional vector $x = (r', p')$, writing the volume element as $d^6x$, and changing to $x^2$ as the integration variable after performing the angular integration.]

Determine the number $N_0$ of the atoms in the ground state of the system as a function of $T/T_c$, and plot of the ratio $N_0/N$ versus $T/T_c$. 

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Reading: Landau and Lifshitz, Secs. 148-150, 152-153
Huang, Secs. 14.1-14.4, 16.1-16.4 (suggested)
(b) The ground state wave function for an isotropic oscillator in three dimensions is

\[ \psi_0(r) = \frac{1}{\pi^{3/4} r_0^{3/2}} e^{-r^2/2r_0^2}, \quad r_0 = \sqrt{\frac{\hbar}{m\omega}}. \]

Determine the scaled number distribution \( \frac{1}{N} \frac{dN}{dr^3(r/r_0)} \) of atoms in the trap and plot it as a function of \( r/r_0 \) for \( N_0/N = 0, 0.1, 0.3, 1.0 \) and \( 0 \leq r/r_0 \leq 10 \). Evaluate \( d^3N/dr^3 \) at \( r = 0 \). Is this a high density? Explain. Compare your results qualitatively with those in the references above. [Hint: the momentum integral which appears must be evaluated numerically. Express the integrand in terms of \( \hbar\omega/kT_c \), \( T/T_c \), and \( r/r_0 \) to see its structure before doing the integration. Note that for a given \( T_c \), \( N = N_c \).]

LD 40: The Bragg-Williams approximation for the 3-dimensional Ising model as a Landau-Ginzburg mean field theory.

Mean field theory (or the Bragg-Williams approximation) gives the expression

\[ m = \mu_0 \tanh \left( \frac{\mu_0 \mathcal{H}}{kT} + \frac{T_c m}{T \mu_0} \right) \]

for the magnetic moment per spin in an Ising model with particles with magnetic moment \( \mu_0 \). \( \mathcal{H} \) is the applied magnetic field. The total magnetization per unit volume is \( M = nm \), where \( n \) is the density of spins.

(a) Show that the magnetic Gibbs function has the form assumed in the Landau-Ginzburg mean field approach when calculated to order \( m^4 \) with \( \mathcal{H} \) retained as a free variable. [Hint: rewrite the relation above as an equation for \( \mathcal{H} \), expand the inverse hyperbolic tangent which appears to order \( m^3 \), and determine \( G_M = F - M\mathcal{H} \). Do not eliminate \( \mathcal{H} \) in \( G_M \).]

(b) Show that \( T_c \) is in fact the critical temperature at which spontaneous magnetization appears at \( \mathcal{H} = 0 \) in the mean field theory. Determine the temperature dependence of the magnetization and the magnetic susceptibility \( \chi = \partial M/\partial \mathcal{H} \) for \( \mathcal{H} \rightarrow 0 \) to lowest order in \( |T_c - T| \) for \( T > T_c \) and for \( T < T_c \), and find the critical exponents \( \beta \) and \( \gamma \). [Hint: minimize \( G_M \) at fixed \( \mathcal{H} \) to determine \( M \). Recall that, in the Landau approach, one is dealing with a Taylor series expansion in \( T \) as well as \( M \), and replace \( T \) by \( T_c \) wherever it does not appear in the difference \( |T - T_c| \). Show that your solution for \( M \) actually minimizes \( G_M \) in the two temperature regions.]

(c) Plot separate figures giving the approximate expressions for \( M/n\mu_0 \) and \( \chi kT_c/n\mu_0^2 \) for \( 0 \leq T/T_c \leq 2 \) and values in the the ranges \( 0 \leq M/n\mu_0 \leq 2 \) and \( 0 \leq \chi kT_c/n\mu_0^2 \leq 4 \). Include on the same figures the exact results for these quantities obtained by solving the equation above and the corresponding equation for \( dM/d\mathcal{H} \) (evaluated for \( \mathcal{H} = 0 \)). [Hint: To find the nontrivial solution for \( M \) numerically for \( T < T_c \), find the zero of \( 1 - \frac{1}{x} \tanh \frac{T_c}{T} x \), \( x = m/\mu_0 \).]
LD 41: First order phase transitions in the Landau theory.

Consider a mean field theory with order parameter $m$ and Landau free energy

$$G = \frac{1}{2}am^2 - \frac{1}{3}bm^3 + \frac{1}{4}cm^4, \quad a, b, c > 0.$$ 

Sketch the possible shapes for $G(m)$ as $b$ is increased from zero with $a$ and $c$ fixed, and show that a first order phase transition occurs for $b$ equal to a critical value $b_c$. Determine $b_c$ and the change in the equilibrium value of $m$ at the transition. [Hint: The scaling $m = (2a/c)^{1/2}m'$, $G = (a^2/c)G'$ is useful for seeing the structure of $G$. Make your sketches for $m'$ small. To find $b_c$, determine when $G$ has two quadratic minima with $G = 0$, but at different values of $m$.]

LD 42: Critical exponents from a Landau free energy.

The Landau free energy for a hypothetical system is of the form

$$F(t, m, h) = -hm + \frac{1}{4}atm^4 + \frac{1}{6}bm^6, \quad a, b > 0,$$

where $m$ is the order parameter, $t = (T - T_c)/T_c$, and the external field $h$ is the variable conjugate to $m$. Show that there is a “second order” phase transition at $T = T_c$ for $h = 0$. Determine the critical exponents $\alpha, \beta, \gamma, \delta$, and show directly that

$$\alpha + 2\beta + \gamma = 2, \quad \delta = 1 + \frac{\gamma}{\beta}.$$

Show explicitly that your solution for $m$ actually minimizes $F$ for $t > 0$ and $t < 0$. Keep only the leading behavior in $t$ for $t \to 0$. [Hint: Calculate the thermodynamic derivatives which determine $S$ and $\chi = \left(\frac{\partial h}{\partial m}\right)_t^{-1}$, and evaluate them for the value of $m$ that minimizes $F$. $C = T^{\frac{\alpha S}{\beta T}}$. To determine $\delta$, minimize $F$ on the critical isotherm $t = 0$ for $h \neq 0$.]