PHYSICS 715

Problem Set 3 **Due Friday, February 10, 2006**

Reading: Landau and Lifshitz, Secs. 32–46

LD 8: Particle distributions in high-energy collisions *via* Gibbs' approach.

The production of large numbers of particles in high-energy collisions has important statistical aspects. Suppose that $N \gg 1$ pions are produced in a proton-proton collision. The pions can be treated as massless at high energies, with individual energies $E = \sqrt{p_\perp^2 + p_\parallel^2}$ in natural units with $c = 1$. Here $\mathbf{p}_\perp$ and $p_\parallel$ are the components of the pion's momentum perpendicular and parallel to the beam direction. The average values $\overline{E}$, $\overline{p_\perp} = \overline{|\mathbf{p}_\perp|}$ of the energy and the transverse momentum per particle are known.

(a) *Derive* the appropriate Gibbs distribution for the particle momenta starting with the Gibbs entropy and assuming that the intrinsic single-particle density in momentum space is the relativistically invariant density $d^3p/E = p_\perp dp_\perp dp_\parallel d\phi/E$ rather than d^3p. where ϕ is the angle around the beam direction. Assume that the particles can be treated as independent (no correlations). You do not need to determine the new constants that appear.

(b) It is customary in reporting experimental results to replace the hard-to-measure variable $p_\parallel$ by the "pseudorapidity" η defined by $\cosh\eta = 1/\sin\theta$ (angles are easy to measure), where $p_\parallel = p_\perp \sinh\eta$, $E = p_\perp \cosh\eta$. Obtain the particle number distribution $dN/d\eta$ as a function of η alone, and show that it can be written as

$$\frac{dN}{d\eta} = \frac{2\pi N/Z}{(\alpha + \beta\cosh\eta)^2}.$$

Obtain explicit formulas, expressed as integrals over η, that would allow you to determine the constants Z, α, and β from known quantities.

The statistical parameter β is naturally identified with a "partition temperature", $\beta = 1/kT_p$, but this is *not* to be interpreted as an equilibrium temperature. In an equilibrium situation, the momentum-space density is just d^3p; the extra factor $1/E$ is appropriate when particles are radiated, and its inclusion represents a dynamical assumption. For the original application of these ideas, see T.T. Chou, C.N. Yang, and E. Yen, Phys. Rev. Lett. **54**, 510 (1985).

LD 9: Anharmonic perturbations in a classical solid.

The Hamiltonian for N particles oscillating around their equilibrium sites in a cubic lattice is

$$H = \sum_{i=1}^{N} H_i, \quad H_i = \frac{p_i^2}{2m} + \tfrac{1}{2}m\omega^2 q_i^2 + \lambda q_i^4.$$

(a) Determine the single-particle partition function $Z_1^{(0)}$ for the system when $\lambda = 0$ so the anharmonic terms are absent. Use derivatives of $Z_1^{(0)}$ to evaluate the averages $\langle q_i^{2n}\rangle_0$ over the unperturbed ($\lambda = 0$) distribution for arbitrary n.

(b) Calculate the ratio of single-particle partition functions $Z_1/Z_1^{(0)}$ correct to second order in λ assuming that $\lambda q^4/kT$ is small on the average, $\langle \lambda q^4/kT\rangle_0 \ll 1$. Use the result to determine the Helmholtz free energy F, the total energy E, the specific heat C, and the entropy S of the N-oscillator system correct to second order in λ. [Hint: expand the expressions for Z_1 and $\ln Z_1$ in powers of λ, and use the expanded results, taken to order λ^2, in the subsequent calculations.]

Note that E *decreases* even though the addition to the potential is positive. How is this possible? Is the change in the entropy expected? Explain. How could the presence of the anharmonic term in the potential be detected in practice? Finally, for what range of temperatures is the calculation valid? Give a quantitative criterion in terms of λ, ω,and m, and explain what it means physically in terms of the motion of the unperturbed oscillators.

LD 10: The central temperature of the sun: Reaction rates in the Maxwell-Boltzmann distribution.

The total luminosity (radiative energy output) of the sun is $L_\odot = 3.86\times 10^{33}$ erg/s, almost all of which is produced in the *pp* cycle [$p+p\rightarrow {}^2\mathrm{H}+e^+ +\nu$, $e^+ + e^- \rightarrow \gamma's$, ${}^2\mathrm{H}+p\rightarrow {}^3\mathrm{He}+\gamma$, ${}^3\mathrm{He}+{}^3\mathrm{He}\rightarrow {}^4\mathrm{He}+2p$]. The average energy produced per cycle which goes into radiation is 26.2 MeV. The first step in the cycle limits the rate. The reactions occur in the core of the sun ($r\leq 0.2\,R_\odot$, average density of protons ~ 30 gm/cm^3). $R_\odot = 6.96\times 10^{10}$ cm.

The number of reactions $pp\rightarrow {}^2\mathrm{H}+e^+ +\nu$ per unit volume per second is $dn/dt = \frac{1}{2}n_p^2\langle\sigma v\rangle$, where n_p is the proton number density, v is the relative velocity of the protons, and σ is the reaction cross section,

$$\sigma = \frac{2Se^{-2\pi\eta}}{m_p v^2}\,,\ \eta = \frac{e^2}{\hbar v} = \alpha\frac{c}{v}\,,\ S = 6.5\times 10^{-55}\,\mathrm{cm}^2\mathrm{erg}\,,\ \alpha\approx 1/137$$

in the customary cgs units. The average is over the normalized *two-particle* Maxwell-Boltzmann momentum distribution. The counting of pairs is already included in the factor $\frac{1}{2}n_p^2$ in dn/dt. Estimate the average temperature of the solar core.

Hints: Use relative and center-of-mass variables $\mathbf{p} = \frac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2)$, $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$ and integrate first over the center-of-mass variables. The remaining integrand in $\langle\sigma v\rangle$ is sharply peaked as a function of p, the magnitude of the relative momentum, as a sketch will show. Determine the value of the momentum p_m at the peak, and change to p/p_m as a new variable to simplify the algebra. Finally, expand $\phi(p)$, the logarithm of the integrand, to second order around its value at the peak and estimate the integral using the resulting Gaussian approximation for the integrand. It will be helpful in getting the final numbers to express the temperature in units of 10^6K, $T = T_6\times 10^6$, identify the most important term in the expression for $L_\odot$, and find T_6 by iteration.

For an original reference, see Critchfield and Bethe, Phys. Rev. **54**, 248 (1938), a fairly early paper in Bethe's Nobel Prize work on stellar energy. J. Bahcall and R. Ulrich, Rev. Mod. Phys. **60**, 297 (1988), Sec. V, give the results of a modern calculation.