

PHYSICS 715 PROBLEM SET 2 SOLUTIONS

1. Still to come.

2. The atom ${}^3\text{He}$ has nuclear spin $1/2$ and is therefore a fermion. Collections of these atoms remain liquid to absolute zero. The density is 0.081 gm/cm^3 .

(a) Calculate the Fermi velocity v_F , the Fermi energy ε_F , and the Fermi temperature T_F .

$$N = 2 \frac{V}{(2\pi)^3} \int_{k < k_F} d^3k = 2 \frac{V}{(2\pi)^3} \frac{4\pi k_F^3}{3} = \frac{V k_F^3}{3\pi^2} \Rightarrow \rho = mn = \frac{m k_F^3}{3\pi^2} \quad (1)$$

and

$$k_F = \left(\frac{3\pi^2 \rho}{m} \right)^{1/3} = \left(\frac{3\pi^2 \times 0.081 \text{ gm/cm}^3}{4 \times 1.66 \times 10^{-24} \text{ gm}} \right)^{1/3} = 0.78 \times 10^8 / \text{cm} \quad (2)$$

$$v_F = \frac{\hbar k_F}{m} = \frac{1.05 \times 10^{-27} \text{ erg} \cdot \text{s} \times 0.78 \times 10^8 / \text{cm}}{4 \times 1.66 \times 10^{-24} \text{ gm}} = 0.12 \times 10^5 \text{ cm/s} \quad (3)$$

$$\varepsilon_F = \frac{m v_F^2}{2} = \frac{4 \times 1.66 \times 10^{-24} \text{ gm} \times (0.11 \times 10^5 \text{ cm/s})^2}{2} = 4.8 \times 10^{-16} \text{ eV} \quad (4)$$

$$T_F = \varepsilon_F / k_B = 4.8 \times 10^{-16} \text{ erg} / 1.38 \times 10^{-16} (\text{erg/K}) = 3.5 \text{ K}$$

(b) Calculate the specific heat at low temperatures and compare to the experimental value $C_V = 2.89 NkT$. Comment on any discrepancy.

$$\frac{C_V}{Nk} = \frac{\pi^2 kT}{2 \varepsilon_F} = \frac{\pi^2 T}{2 T_F} = 1.4 T(K). \quad (5)$$

This is a numerical formula valid if temperature is measured in degrees K. The specific heat is enhanced by a factor of $2.89/1.4 \approx 2$ by the interactions of the helium atoms. The interaction acts to slow the atoms down, since they get into each other's way. This effectively increases the mass, which increases the specific heat.

3. Let g represent the spin degeneracy. $g = 4$ for spin 3/2 and $g = 2$ for spin 1/2. Then

$$U(g, T = 0) = g \frac{V}{(2\pi)^3} \int_{k < k_F(g)} \frac{\hbar^2 k^2}{2m} d^3 k = g \frac{V}{(2\pi)^3} \frac{\hbar^2}{2m} 4\pi \int_0^{k_F(g)} k^4 dk = g \frac{V \hbar^2 k_F^5(g)}{20\pi^2} \quad (6)$$

and

$$P(g, T = 0) = \frac{dU(g, T = 0)}{dV} = g \frac{k_F^5(g)}{20\pi^2}. \quad (7)$$

Now the two pressures are equal, so

$$1 = \frac{P(4)}{P(2)} = \frac{4k_F^5(4)}{2k_F^5(2)} \Rightarrow \frac{k_F(4)}{k_F(2)} = 2^{-1/5} \quad (8)$$

while

$$n(g, T = 0) = g \frac{1}{(2\pi)^3} \int_{k < k_F(g)} d^3 k = g \frac{1}{(2\pi)^3} 4\pi \int_0^{k_F(g)} k^2 dk = g \frac{k_F^3(g)}{6\pi^2}. \quad (9)$$

Hence

$$\frac{n(g = 4, T = 0)}{n(g = 2, T = 0)} = \frac{4k_F^3(4)}{2k_F^3(2)} = 2 \times \left(\frac{k_F(4)}{k_F(2)} \right)^3 = 2^{2/5} = 1.32. \quad (10)$$

At high temperatures the gases become classical: $P = nkT$ independent of spin degeneracy. Hence the densities are equal.

4. In a white dwarf star α -particles form a stationary lattice and the electrons are a degenerate gas. Assume that the temperature is zero. For a white dwarf with mass M and radius R

(a)

$$E_g = \int d^3 r V(\vec{r}) \rho(\vec{r}) = \rho \int d^3 r V(\vec{r}) = 4\pi\rho \int r^2 dr V(r), \quad (11)$$

with

$$V(r) = -GM(r)/r \quad (12)$$

where $M(r)$ is the mass inside a sphere of radius r :

$$M(r) = \frac{4\pi r^3}{3} \rho, \quad (13)$$

so

$$E_g = -4\pi G \frac{4\pi\rho^2}{3} \int_0^R r^4 dr = -\frac{16\pi^2 G \rho^2 R^5}{15} = -\frac{3G}{5R} M^2 \quad (14)$$

(b) Calculate the kinetic energy of the electrons.

$$E_k = \frac{2V}{8\pi^3} 4\pi \int_0^{k_F} \frac{\hbar^2 k^2}{2m_e} k^2 dk = \frac{V \hbar^2 k_F^5}{10\pi^2 m_e} \quad (15)$$

and

$$N_e = \frac{2V}{8\pi^3} 4\pi \int_0^{k_F} k^2 dk = \frac{V k_F^3}{3\pi^2} \quad (16)$$

so

$$\begin{aligned} E_k &= \frac{V \hbar^2}{10\pi^2 m_e} \left(\frac{3\pi^2 N_e}{V} \right)^{5/3} = \frac{(3\pi^2)^{5/3} \hbar^2}{10\pi^2 m_e} (N_e)^{5/3} V^{-2/3} \quad (17) \\ &= \frac{(3\pi^2)^{5/3} \hbar^2}{10\pi^2 m_e} \left(\frac{2M}{m_\alpha} \right)^{5/3} \left(\frac{4\pi R^3}{3} \right)^{-2/3} = \frac{3(6\pi^2)^{1/3} \hbar^2}{5m_e m_\alpha^{5/3}} M^{5/3} R^{-2} \quad (18) \end{aligned}$$

where I have used the fact that

$$N_e = 2N_\alpha = 2M/m_\alpha \quad (19)$$

since the total charge must vanish (or be very small), and $m_e \ll m_\alpha$.

(c) The virial theorem states that the gravitational energy and kinetic energy are about equal in magnitude. Deduce a relation between the mass and the radius of the form $M^x R^y = C$. Determine the exponents x and y and the constant C .

$$\frac{3G}{5R} M^2 = \frac{3(6\pi^2)^{1/3} \hbar^2}{5m_e m_\alpha^{5/3}} M^{5/3} R^{-2} \quad (20)$$

which gives

$$x = 1/3, y = 1, \quad C_{w-d} = \frac{3(6\pi^2)^{1/3} \hbar^2}{G m_e m_\alpha^{5/3}}. \quad (21)$$

Neutron stars consist predominantly of a liquid of neutrons at temperature zero.

(d) Repeat part (c) for neutron stars.

This is the same except that we must replace N_e by N_n and use $M = N_n m_n$, yielding

$$C_n = \frac{3(6\pi^2)^{1/3} \hbar^2}{2^{5/3} G m_n^{8/3}}. \quad (22)$$

It is very rare in physics to find G and \hbar in the same formula.

5. Assume that neutrinos have zero mass and spin 1/2. Consider a universe of volume V and temperature T filled with primordial neutrinos. What is the energy density of the neutrinos?

There are 6 kinds of neutrinos, electron-type, muon-type, and tau-type, and their antiparticles. There is no spin degeneracy because all neutrinos are left-handed and antineutrinos are right-handed.

$$\frac{E}{V} = \frac{6}{(2\pi)^3} \int \frac{1}{e^{\beta(\hbar ck - \mu)} + 1} \hbar ck d^3k = \frac{6\hbar c}{2\pi^2} \int_0^\infty \frac{k^3 dk}{e^{\beta(\hbar ck - \mu)} + 1} \quad (23)$$

It is generally thought that the chemical potential of the neutrinos is much smaller than their temperature. If this is the case, then

$$\begin{aligned} \frac{E}{V} &= \frac{6\hbar c}{2\pi^2} \int_0^\infty \frac{k^3 dk}{e^{\beta\hbar ck} + 1} = \frac{6\hbar c}{2\pi^2(\beta\hbar c)^4} \int_0^\infty \frac{x^3 dx}{e^x + 1} = \frac{6(k_B T)^4}{2\pi^2(\hbar c)^3} \int_0^\infty \frac{x^3 dx}{e^x + 1} \\ &= \frac{7\pi^4}{20} \frac{(k_B T)^4}{2\pi^2(\hbar c)^3} = \frac{7\pi^2}{40} \frac{(k_B T)^4}{(\hbar c)^3} \end{aligned} \quad (24)$$

Note that this is 21/8 of the value for photons at the same temperature. (You still get full marks for this if you just used electron neutrinos.)

6. Show that if \bar{n} is the occupancy of a single quantum state in a Bose system, then

$$\overline{(\Delta n)^2} = \bar{n}(1 + \bar{n}). \quad (25)$$

Find the corresponding formula for a Fermi system.

Given the formulas

$$Q = \sum_{states} e^{-\beta n_i(\varepsilon_i - \mu)}, \quad (26)$$

$$\bar{n}_i = \frac{1}{Q} \sum_{states} n_i e^{-\beta n_i(\varepsilon_i - \mu)} = \frac{-1}{\beta} \frac{\partial \ln Q}{\partial \varepsilon_i} = \frac{1}{e^{\beta(\varepsilon_i - \mu)} - 1} \quad (27)$$

$$\overline{n_i^2} = \frac{1}{Q} \sum_{states} n_i^2 e^{-\beta n_i(\varepsilon_i - \mu)} = \frac{1}{\beta^2 Q} \frac{\partial^2 Q}{\partial \varepsilon_i^2} \quad (28)$$

we find

$$\frac{\partial^2 \ln \mathcal{Q}}{\partial \varepsilon_i^2} = \frac{\partial}{\partial \varepsilon_i} \left(\frac{1}{\mathcal{Q}} \frac{\partial \mathcal{Q}}{\partial \varepsilon_i} \right) \quad (29)$$

$$= -\beta^2 \frac{1}{\mathcal{Q}^2} \left(\frac{\partial \mathcal{Q}}{\partial \varepsilon_i} \right)^2 + \frac{1}{\mathcal{Q}} \frac{\partial^2 \mathcal{Q}}{\partial \varepsilon_i^2} \quad (30)$$

$$= -\beta^2 \overline{n_i^2} + \beta^2 \overline{n_i^2} = \beta^2 (\overline{\Delta n_i})^2. \quad (31)$$

However,

$$\frac{\partial^2 \ln \mathcal{Q}}{\partial \varepsilon_i^2} = \frac{\partial}{\partial \varepsilon_i} \left(\frac{\partial \ln \mathcal{Q}}{\partial \varepsilon_i} \right) = -\beta \frac{\partial}{\partial \varepsilon_i} n_i = -\beta \frac{\partial}{\partial \varepsilon_i} \frac{1}{e^{\beta(\varepsilon_i - \mu)} - 1} \quad (32)$$

$$= \frac{\beta^2 e^{\beta(\varepsilon_i - \mu)}}{[e^{\beta(\varepsilon_i - \mu)} - 1]^2} = \beta^2 \overline{n_i^2} \left(\frac{1}{\overline{n_i}} + 1 \right) = \beta^2 \overline{n_i} (\overline{n_i} + 1). \quad (33)$$

Comparing these two formulas, we find

$$(\overline{\Delta n_i})^2 = \overline{n_i} (\overline{n_i} + 1). \quad (34)$$

Doing precisely the same derivation for fermions means only replacing $1/[e^{\beta(\varepsilon_i - \mu)} - 1]$ by $1/[e^{\beta(\varepsilon_i - \mu)} + 1]$, and we find

$$\frac{\partial^2 \ln \mathcal{Q}}{\partial \varepsilon_i^2} = \frac{\beta^2 e^{\beta(\varepsilon_i - \mu)}}{[e^{\beta(\varepsilon_i - \mu)} + 1]^2} = \beta^2 \overline{n_i^2} \left(\frac{1}{\overline{n_i}} - 1 \right) = \beta^2 \overline{n_i} (1 - \overline{n_i}) \quad (35)$$

and

$$(\overline{\Delta n_i})^2 = \overline{n_i} (1 - \overline{n_i}). \quad (36)$$

8.

(a) We can write

$$\frac{1}{V} \ln \mathcal{Q} = -\frac{2\pi}{h^3} \int_0^\infty dp p \ln \left(1 - z e^{-\beta p^2/2m} \right) - \frac{1}{V} \ln (1 - z) \quad (37)$$

We consider the behavior of the chemical potential for a system at fixed number density n . If μ (which must always be negative for a Bose system) is a smooth function of temperature, then there is no Bose condensation.

$$n = \frac{1}{(2\pi)^2} \int \frac{1}{e^{\beta(\hbar^2 k^2/2m - \mu)} - 1} d^2 k = \frac{1}{(2\pi)^2} 2\pi \int_0^\infty \frac{k dk}{e^{\beta(\hbar^2 k^2/2m - \mu)} - 1} \quad (38)$$

$$= \frac{1}{2\pi} \frac{m}{\beta \hbar^2} \int_0^\infty \frac{dx}{e^{x-y} - 1} = \frac{m}{2\pi \beta \hbar^2} \int_{-y}^\infty \frac{dx}{e^x - 1} \quad (39)$$

where $x = \beta \hbar^2 k^2/2m$ and $y = \beta \mu$. We may further simplify by writing $u = e^{-x/2}$:

$$\begin{aligned} \frac{2\pi \hbar^2 n}{m} &= k_B T \int_{-y}^\infty \frac{e^{-x/2} dx}{e^{x/2} - e^{-x/2}} = 2k_B T \int_0^{e^{y/2}} \frac{du}{1/u - u} = -2k_B T \int_0^{e^{y/2}} \frac{u du}{u^2 - 1} \quad (40) \\ &= -k_B T \ln |u^2 - 1|_0^{e^{y/2}} = -k_B T \ln |e^{\mu/k_B T} - 1|. \end{aligned}$$

Defining $T_0 = 2\pi^2 \hbar^2 n / m k_B$, we solve for μ and find

$$\mu = k_B T \ln(1 - e^{-T_0/T}). \quad (41)$$

This function varies smoothly from $\mu(T = 0) = 0$ to $\mu(T = \infty) = -\infty$. Hence there is no condensation.