1 The Big Bang

The big bang theory involves two basic assumptions. The first is that the solution to the equations above do indeed go back to the initial condition that $R(t = 0) = 0$, or nearly so. Since all such solutions have $\rho(R) \sim R^{-n}$ with $n > 0$, as in the example, the energy density of the universe approaches infinity as $t \to 0$. Second, the universe was in thermodynamic equilibrium at early times, say for $t > 10^3$ ms. The second assumption is just as important as the first, and it allows us to apply our knowledge of thermodynamics to the problem.

Because of the high energy density, the temperatures are also extremely high in the early universe. As it expands the temperature drops. Let us first consider what we know about the energy levels of such systems. Here we take a clue from chemical reactions. Consider a system with two kinds of particles $A$ and $B$ that can form a compound $AB$ with binding energy $\varepsilon$. If $k_B T << \varepsilon$, then all particles will bind if the concentrations of $A$ and $B$ are equal. Otherwise all will bind except for those left over. If $k_B T >> \varepsilon$, then the equilibrium concentration of the compound will be very low and most of $A$ and $B$ will be free, because of the higher entropy of two particles being able to move around instead of just one.

Note that there is a very important aspect of this temperature evolution, namely that the total number of the two types of particle is conserved. To specify a problem of this kind completely we must specify the values of conserved quantities. Thermodynamics can take care of the rest. Unfortunately, this means we need more assumptions in our theory, the values of several conserved quantities: total charge, total baryon number, and total electron, muon, and tau lepton numbers. Those we will try to guess or deduce later.

Just these few simple assumptions give us enough to sketch the early history of the universe, measuring time by the temperature for the moment. All we need to do is to tabulate the various interesting energies $\varepsilon$ from particle physics and ask what happens as the temperature passes through the corresponding value $T = \varepsilon/k_B$. Keep in mind that when we speak of energies now we must also include rest-mass energies of the particles, in contrast to a laboratory chemical system. The only number we need is $1 \varepsilon V$ corresponds roughly to $10^4 K$.

0. Asymptotic freedom. At very high temperatures, the strong, weak, and electromagnetic interactions all have comparable strength, and all become very weak due to the phenomenon of asymptotic freedom. All particles are equally numerous at these early times. It is in the transition from this era to the next where matter and antimatter may get out of balance, but that is not part of this course.

1. Free Quarks $10^{12} K < T < 10^{32} K$: $k_B T$ is above the rest mass ($\sim 1 GeV \sim 10^9 eV$) of neutrons, protons, pions, and other hadrons. We do not know what the equation of state would be when these particles are very numerous and so close together that the strong interaction dominates. Hence this era of the universe remains relatively mysterious, the Dark Ages,
and is currently the subject of intense research. Thermodynamics remains very relevant to this period. The broken symmetry (nonzero vacuum expectation value of the Higgs field) that is a fundamental feature of the standard model is probably the result of a phase transition at these very early times. Also not part of this course, (which is too bad, because it is very similar to the Landau theory of phase transitions). All particles are still equally numerous times

2. $T = 3 \times 10^{11} - 10^{12} K : k_B T$ has dropped below the rest mass of hadrons but remains well above the rest mass of the lighter particles ($\sim 1 \text{ MeV} \sim 10^{6} \text{eV}$ for the electrons, for example). In this period there are only few protons and neutrons, in equal numbers. But they are very scarce compared to the photons, leptons, antileptons, neutrinos and antineutrinos. The reason for this is that all the baryons and antibaryons have annihilated except for the small number of baryons that we see left over today, while the total number of photons or leptons + antileptons is not conserved and there is plenty of energy available to make them in pairs. Reactions such as $\mu^+ + \mu^- \leftrightarrow 2 \gamma$, $e^+ + e^- \leftrightarrow 2 \gamma$, $n + \nu_e \leftrightarrow p + e^-$ and $n + \nu_{\mu} \leftrightarrow p + \mu^-$ are taking place very rapidly, keeping all the particles in equilibrium.

3. $T = 1 \times 10^{11} - 3 \times 10^{11} K : k_B T$ is now comparable to $2 m_\mu$, so the first reaction above starts going to the right but not to the left, and the muons disappear. Because the muon-type neutrinos cannot couple to anything but muons, they fall out of equilibrium with the rest of the matter in the universe. They have essentially not interacted with the rest of the universe since then. A similar thing happens later to the electron-type neutrinos. It has already happened to the tau neutrinos (mass $\sim 1 \text{ GeV}$), but way back in the Dark Ages.

4. $T = 5 \times 10^9 K - 10^{11} K :$ the interesting energy in this range is the neutron-proton mass difference. The reaction $n + \nu_e \leftrightarrow p + e^-$ starts to go to the right more than to the left, meaning that the protons start to become more numerous than the neutrons.

5. $T = 1 \times 10^9 K - 5 \times 10^9 K : k_B T$ is now comparable to $2 m_e$, so the reaction $e^+ + e^- \leftrightarrow 2 \gamma$ now starts going mostly to the right, leaving only the relatively small number of electrons that we see today. The reactions that interconvert protons and neutrons such as $n + \nu_e \leftrightarrow p + e^-$ now essentially stop. Numerical calculations indicate that the final value of the ratio of protons to neutrons should be about 1:5, which is pretty much what we see today. Primordial nucleosynthesis can now take place. Neutrons and protons fuse to form the very stable compound nucleus $^4\text{He}$ as well as very small amounts of the less stable $d$, $^3\text{He}$, and $^7\text{Li}$. We are left with about 27% $^4\text{He}$ by weight compared to $H$.

6. $T = 1 \times 10^9 K - 4 \times 10^9 K :$ Electromagnetic interactions between the the photons, electrons, and nuclei keep these particles in thermal equilibrium.

7. $T = 4 \times 10^9 K :$ the protons and electrons combine to form neutral hydrogen and helium, which then falls out of equilibrium with the photons, which do not interact with the rest of the universe again.
8. $T = 4 \times 10^3 K$-today: we are finally left with remnant backgrounds of photons and neutrinos as well as the bits of matter that were left over when all the antimatter was annihilated. Matter is nearly uniformly distributed at the beginning of this stage. But now the dominant interaction is gravitational, which causes clumping and then galaxy formation at larger scales, and star formation on smaller scales.

1.0.1 Time development of the early universe

At early times, $dR/dt$ was large and we may neglect $k$ in the Einstein equation. This gives

$$\left(\frac{dR}{dt}\right)^2 = \frac{8\pi G}{3} \rho R^2$$

(1)

for the dynamical equation of the universe. We have already seen that if $P << \rho$, as is true today, then $\rho \sim R^{-3}$. If the energy of the universe is in the form of radiation, or of extremely relativistic particles, then $P = \rho/3$. Then we have

$$\frac{d}{dR} (\rho R^3) = -\rho R^2,$$

(2)

which says that $\rho \sim R^{-4}$. Hence, in the early universe with $R$ very small, the energy density was dominated by radiation and extremely relativistic particles. The crossover occurs somewhere around $10^4 K$, as it happens, so most of our discussion will be concentrated in the era when $\rho \sim R^{-4}$.

To get a very rough idea of the evolution of the early universe, let us suppose that all the energy is in this form, and all of the components of it are in equilibrium. We already know that $\rho \sim T^4$ for radiation, so this tells us that $T \sim 1/R$. Substituting back into the Einstein equation, we also find that $R \sim t^{1/2}$ and $T \sim t^{-1/2}$.

When $P$ is comparable to $\rho$, we say that the universe is "radiation-dominated". When $P << \rho$, we say that the universe is "matter-dominated". The early universe is radiation-dominated. The present universe is matter-dominated.

1.0.2 Statistical mechanics of particles

Our aim now is to give some detailed justification for the history of the early universe sketched above.

The number density of particles of type $i$ having momentum between $p$ and $dp$ is

$$n_i(p) = 4\pi g_i h^{-3} p^2 \frac{1}{e^{\beta(\epsilon - \mu_i)} \pm 1},$$

(3)

where $\epsilon^2 = m_i^2 + p^2$ and $\mu_i$ is the chemical potential of the $i$th species. $g_i$ is the spin degeneracy. For photons, $\mu_\gamma = 0$, while for particles $P$ and antiparticles $\bar{P}$ we must have $\mu_P = -\mu_{\bar{P}}$, since they can always annihilate into photons.
and the total chemical potential is conserved in all such reactions. Similar considerations applied to weak interactions give constraints such as:

\[
\mu_{e^-} - \mu_{\nu_e} = \mu_n - \mu_p = \mu_{\mu^-} - \mu_{\nu_\mu} = \mu_{e^-} - \mu_{\nu_e},
\]

leaving only five independent chemical potentials, one each for the proton, electron, and the three neutrinos. These can only be fixed by choosing the five conserved quantities: charge density, baryon number density, electron, muon, and tau lepton number density. Since all of these are conserved, they vary as \( R^{-3} \), and may be taken to be small in the radiation-dominated phase. For electron-lepton number we can write

\[
n_e(p) = n_{e^-}(p) + n_{e^+}(p) = 8\pi g_h h^{-3} p^2 \left[ \frac{1}{e^{\beta(\epsilon - \mu_{e^-})} + 1} + \frac{1}{e^{\beta(\epsilon + \mu_{e^-})} + 1} \right]
\]

\[
> > n_{e^-}(p) - n_{e^+}(p) = \frac{1}{e^{\beta(\epsilon - \mu_{e^-})} + 1} - \frac{1}{e^{\beta(\epsilon + \mu_{e^-})} + 1},
\]

so we have \( \mu_{e^-} \approx \mu_{e^+} \ll m_e \), and similarly for all other particles. So the density is given by

\[
n_i(p, m) = 4\pi g_h h^{-3} p^2 \left[ \frac{1}{e^{\beta(m^2 + p^2)} + 1} \right].
\]

and we see immediately that if \( m/k_B T >> 1 \), then the concentration of particles of mass \( m \) is negligible. In general, as the temperature increases through the appropriate value, there is a crossover from very low concentration at low temperatures to a high concentration given by the relativistic law:

\[
n_i(p, m = 0) = 4\pi g_h h^{-3} p^2 \frac{1}{e^{\beta p} + 1}.
\]

Integrating this over \( p \) gives \( n_i \approx (k_B T / h)^3 \) and \( \rho \approx k_B T (k_B T / h)^3 \), laws valid for \( k_B T >> m \). This justifies the overall picture of particle concentrations in the above history.

Now what about the statement about neutrinos going out of equilibrium? For \( T < 1.5 \times 10^{12} K \), we have essentially only muons, electrons, photons, and neutrinos in the universe. The cross-sections for neutrino scattering and production in processes such as

\[
\nu_\mu + \mu^- \leftrightarrow \nu_\mu + e^{-}
\]

is given by the weak interaction expression

\[
\sigma = g_w^2 h^{-4} (k_B T)^2.
\]

\( g_w = 1.4 \times 10^{-49} \) erg-cm\(^3\) is quite accurately known since it determines the muon lifetime \( t_\mu = 2.19 \mu s \). (This short lifetime coming from the reaction
\( \mu^+ \rightarrow e^+ + \nu_e + \bar{\nu}_\mu \) also explains why there are no muons left in today’s universe.) So the scattering rate is

\[ \sigma n_\mu \sim g_\mu^2 \hbar^{-7} (k_B T)^5 \exp(-m_\mu/k_B T) \]  

(11)

when \( k_B T > 2m_\mu \). The decoupling temperature \( T_D \) is determined by the scattering rate becoming comparable to the expansion rate:

\[ \sigma n_\mu = \frac{1}{R} \frac{dR}{dt} \]  

(12)

but the dynamical equation

\[ \left( \frac{dR}{dt} \right)^2 = \frac{8\pi G}{3} \rho R^2 \]  

(13)

can be rewritten as

\[ \frac{1}{R} \frac{dR}{dt} = \left( \frac{8\pi G}{3} \rho \right)^{1/2} = \left[ \frac{8\pi G}{3} k_B T(k_B T/\hbar)^3 \right]^{1/2} \]  

(14)

and so

\[ g_\mu^2 \hbar^{-7} (k_B T_D)^5 \exp(-m_\mu/k_B T_D) = \left[ \frac{8\pi G}{3} k_B T(k_B T/\hbar)^3 \right]^{1/2} \]  

(15)

Inserting numerical values gives

\[ \exp(-10^{12} K/T_D) = (10^{10} K/T)^3 \]  

(16)

which gives \( T_D = 1.3 \times 10^{11} K \). Note that once the temperature has cooled through \( T_D \) the scattering rate will begin to decrease exponentially, since the muon density is decreasing exponentially. Thus the decoupling is rapid and complete.

In the time since, the neutrinos have evolved according to the equation derived above for radiation: \( T \sim 1/R \). Hence today there should be a neutrino background with a temperature given by \( T_{0} = T_D \times [R(T_D)/R_0] \). We will return to the numerical value of this temperature below.

Now we are ready to plot the overall history in terms of the more familiar variable of time rather than temperature. In the time between \( T = 10^{12} K \) and \( T = 5 \times 10^9 K \), we have

\[ \rho_\nu = 4\pi \hbar^{-3} \int_0^\infty p^3 \frac{1}{e^{\beta q} + 1} dp = \frac{7\pi^5}{30\hbar^3} (k_B T)^4 = \frac{7}{16} \alpha T^4 , \]  

(17)

where I have defined \( \alpha = (8\pi^5/15)k_B^4 \hbar^{-3} \). Similarly \( \rho_{e^-} = \rho_{\bar{\nu}_\mu} = 2\rho_\nu = 7\alpha T^4 /8 \).

Also

\[ \rho_\nu = 4\pi \hbar^{-3} \int_0^\infty p^3 \frac{1}{e^{\beta q} - 1} dp = \alpha T^4 . \]  

(18)
Counting 4 types of neutrinos, electrons, positrons, and photons, we have

\[ \rho = \frac{9}{2} a T^4. \]  

(19)

The dynamical equation is

\[ \left( \frac{dR}{dt} \right)^2 = \frac{8 \pi G}{3} \rho R^2 \]  

(20)

which we may also write as

\[ \frac{1}{R} \frac{dR}{dt} = \left( \frac{8 \pi G}{3} \rho \right)^{1/2} \]  

(21)

or, since \( \rho \sim R^{-4} \), this is

\[ \frac{1}{\rho} \frac{d\rho}{dt} = -4 \left( \frac{8 \pi G}{3} \rho \right)^{1/2} \]  

(22)

whose solution is

\[ t = \left( \frac{3}{32 \pi G \rho} \right)^{1/2} + \text{const.} = \left( \frac{1}{48 \pi G a T^4} \right)^{1/2} + \text{const.} \]  

(23)

Taking \( t = 0 \) at the singular point \( T \to \infty \) and putting in numerical values gives

\[ t = 1.1 s \times \left( \frac{T}{10^{10} K} \right)^{-2} \]  

(24)

So to go from \( 10^{12} K \) to \( 10^{11} K \) takes 10ms, while to go from \( 10^{11} K \) to \( 10^{10} K \) takes about a second. While the formula breaks down during the electron-positron recombination era starting at \( t = 1 s \), it still gives a rough rule of thumb, and we can say that to get to \( T = 10^9 K \) took about \( 10^4 s \), which is several hours. Continuing down to \( T = 4000 K \), we find that the universe was about \( 4 \times 10^5 yrs \) old when the nuclei and electrons recombined.

The entropy density of the electrons, positrons, and photons when \( T > 5 \times 10^9 K \) was

\[ s = \frac{4 R^3}{3T} (\rho_{e^-} + \rho_{e^+} + \rho_{\gamma}) = \frac{11}{3} a R^3 T^3, \]  

(25)

while after the disappearance of the electrons and positrons by about \( T = 10^9 K \) we have

\[ s = \frac{4 R^3}{3T} \rho_{\gamma} = \frac{4}{3} a R^3 T^3, \]  

(26)
and since this annihilation was an adiabatic process, the entropy remained constant. So the effect was to heat the photons, increasing $RT$ (otherwise a constant) by a factor of $(11/4)^{1/3} \approx 1.4$ in this period. This is the only real deviation from the $T \sim 1/R$ in the early history of the universe. The neutrinos were not heated by this process since they had already decoupled. They continued to obey $T_\nu \sim 1/R$. Hence the temperature of the photons exceeds that of the neutrinos by a factor of 1.4. This should still be true today, so big bang cosmology makes the wonderful prediction that there should be neutrino background radiation at a temperature of about $2.73 K/(11/4)^{1/3} = 1.95 K$. Unfortunately, no one has so far figured out a way to detect it.

1.1 Helium Synthesis

One of the most remarkable predictions of big bang cosmology is the present abundance of helium, which is almost entirely in the form of $^4\text{He}$. The matter in the universe essentially consists of about 75% hydrogen and 25% helium, by weight. All the other elements make up only a small fraction. This abundance of helium would be a great puzzle without big bang cosmology, since the helium fraction is much too much to have been produced in stars, unlike the small amounts of heavy elements. So let us see how much would have been produced in the early universe.

When $T > 10^{12} K$, the number of neutrons and protons is almost equal. There are very slightly more protons because of the mass difference $\Delta m = m_n - m_p = 1.293 \text{ MeV}$, which corresponds to a temperature of $T = 1.29 \text{ MeV}/k_B = 1.4 \times 10^{10} K$. As time goes on, the protons will become more common because of their smaller mass.

The heavier nuclei are still very short-lived when $T > 10^{12} K$, but $n$ and $p$ interconvert because of the weak interactions

$$n \leftrightarrow p + \ell^- + \bar{\nu}_\ell \quad (27)$$

$$n + \nu_\ell \leftrightarrow p + \ell^- \quad (28)$$

$$n + \ell^+ \leftrightarrow p + \bar{\nu}_\ell. \quad (29)$$

The rates for all of these reactions are calculable within the standard model. The equations for the rates are somewhat complex, but we can give an example that illustrates the main factors that come in. The second reaction proceeds to the right at a rate (per neutron) of

$$R_{n+\ell^+ \rightarrow p+\bar{\nu}_\ell} = C \int E_{\nu_\ell}^2 p_{\ell}^2 f(E_\ell)[1 - f(E_{\nu_\ell})] \, dp_{\ell}, \quad (30)$$

where $E_{\nu_\ell}^2 p_{\ell}^2$ is a phase space factor. $f(E_\ell)$ is the Fermi function for the leptons $\ell^+$. It must be there because the reaction is possible only if the lepton is there. $[1 - f(E_{\nu_\ell})]$ is the probability that the final neutrino state is unoccupied. If it
is already occupied, then the reaction certainly cannot go - the Pauli principle would forbid it. We must take into account the fact that the temperature of the neutrinos may be slightly different from that of the leptons. $C$ is a constant proportional to the square of the appropriate weak coupling: $C = (g_\nu^2 + g_A^2)/2\pi^3\hbar^2$, and $g_\nu = 1.42 \times 10^{-39} \text{erg/cm}^3$ and $g_A = 1.18 g_\nu$. Similar expressions can be written down for the five other reactions for various $\ell$, counting going to the left and the right as different, and for the different lepton species, giving 18 total reactions.

Focusing now only on the $n$ and $p$, we can write total rates

$$R_{n\rightarrow p} = \sum_{\ell = e, \mu, \tau} (R_{n \rightarrow p + \ell^- + \bar{\nu}_\ell} + R_{n + \nu_\ell \rightarrow p + \ell^- + \bar{\nu}_\ell})$$

$$R_{n\rightarrow p} = \sum_{\ell = e, \mu, \tau} (R_{p + \ell^- + \bar{\nu}_\ell \rightarrow n} + R_{p + \ell^- \rightarrow n + \nu_\ell} + R_{p + \bar{\nu}_\ell \rightarrow n + \ell^+})$$

The main thing we want to calculate at this point is the ratio of neutrons and protons. At early times when $k_B T >> \Delta m$, we can neglect the mass of the leptons and note that the momentum of the lepton will be roughly equal to that of the neutrino, so we have

$$R_{n\rightarrow p} \sim \int_{-\infty}^{\infty} q^4 (1 + e^{\Delta mc^2})^{-1} [1 - (1 + e^{\Delta mc^2})^{-1}] dq \sim T^5.$$  \hspace{1cm} (33)

We will not calculate the coefficient, other than to say that

$$R_{n\rightarrow p} = C \left( \frac{T}{10^{10} K} \right)^5,$$

\hspace{1cm} (34)

where $C \sim O(1)$. The age of the universe during this era is given by

$$t = 1.1 s \times \left( \frac{T}{10^{10} K} \right)^{-2},$$

\hspace{1cm} (35)

so $R_{n\rightarrow p} t >> 1$ as long as $T >> 10^{10} K$. This means that the reaction proceeds to equilibrium. If we call the neutron and proton fractions $n_n$ and $n_p$, respectively, then we have $n_n + n_p = 1$ and $n_n/n_p = \exp(-\Delta mc^2/k_B T)$, so

$$n_n = \frac{1}{1 + e^{\Delta mc^2/k_B T}},$$

\hspace{1cm} (36)

and

$$n_p = \frac{1}{1 + e^{-\Delta mc^2/k_B T}}.$$ 

\hspace{1cm} (37)

Eventually, the reactions that require 2 or 3 particles in the initial state become slower, and neutron decay $n \rightarrow p + e^- + \bar{\nu}_e$ dominates. The domination becomes complete at about $T = 1.3 \times 10^9 K$, but between $T = 10^{12} K$ ($t < 1 \text{ms}$) and
\[ T = 1.3 \times 10^9 \text{ K} \ (t \approx 20 \text{ s}) \] numerical solution of the rate equations is necessary. After \( t \approx 20 \text{ s} \) we have

\[ n_n = n_n(t = 20 \text{ s}) e^{-(t-20)/t_n} \]  \( (38) \)

where \( t_n \) is the lifetime of the neutron, which is \( t_n = 1013 \text{ s} \). \( n_n(t = 20 \text{ s}) = 0.164 \). The exponential decay continues until actual nucleosynthesis begins. This occurs in steps:

\[ n + p \rightarrow d + \gamma \]  \( (39) \)

\[ d + d \rightarrow ^{3}\text{He} + n \]  \( (40) \)

\[ d + d \rightarrow ^{3}\text{He} + p \]  \( (41) \)

\[ ^{3}\text{He} + d \rightarrow ^{4}\text{He} + n \]  \( (42) \)

and so on. The actual build-up of heavier nuclei is dominated by the following effect. The binding energy of \( d \) is very small, and it is easily broken up. Hence the equilibrium concentration of \( d \) is quite low until \( T \approx 10^6 \text{ K} \ (t \approx 220 \text{ s}) \). Thus \( d - d \) collisions are very rare and there is no formation of heavier nuclei. However, once the concentration of \( d \) builds up, then these nuclei form very quickly. In fact, because of the high binding energy of the \( \alpha \)-particle, essentially all neutrons are bound into \(^{4}\text{He}\). Because of the speed of this sequence of strong interactions compared to the weak decay of the neutron, the amount of helium produced in the early universe is due entirely to the neutrons still remaining at \( t = 220 \text{ s} \). We have

\[ n_n = 0.164 e^{-(220-20)/1013} = 0.135. \]  \( (43) \)

The helium abundance is usually given as the fraction by weight \( Y \) of helium. Essentially all the rest of the mass is \(^1\text{H}\), so we can write

\[ n_n = \frac{N_n}{N_n + N_p} = \frac{2N_{^{4}\text{He}}}{2N_{^{4}\text{He}} + 2N_{^{4}\text{He}} + N_H} = \frac{2N_{^{4}\text{He}}}{4N_{^{4}\text{He}} + N_H}, \]  \( (44) \)

while, since the masses satisfy \( m_{^{4}\text{He}} = 4m_H \), we have

\[ Y = \frac{4N_{^{4}\text{He}}}{4N_{^{4}\text{He}} + N_H} = 2n_n. \]  \( (45) \)

Finally, \( Y = 2n_n = 0.27 \). This value is in good agreement with observations.

Other elemental abundances that are due to cosmological processes are usually quoted as a fraction of hydrogen (proton) abundance. For example, those of deuterium and helium-3 are: \( \frac{D}{H} \approx \frac{^{3}\text{He}}{H} \approx 10^{-5} \), while \( \frac{^7\text{Li}}{H} \approx 10^{-10} \). These nuclei have very much lower binding energy per nucleon than \(^{4}\text{He}\), which accounts for their low densities.

The agreement of theory and observation for light element abundances is very reasonable, considering that they span many orders of magnitude.
1.1.1 Cosmic Microwave Background Radiation

Electromagnetic radiation decoupled from matter when the protons (and a few $\alpha$-particles) and electrons combined into neutral atoms at $T_{\text{dec}} = 4 \times 10^3 \, K$ ($t_{\text{dec}} = 2 \times 10^{12} \, s$). At a stroke, the mean free time for photon scattering and absorptions went from very short to longer. The expansion of the universe since that time has been adiabatic as far as the photons are concerned. Let us look at how a comoving volume $V = 4\pi r^3 / 3$ with constant entropy evolves.

$$0 = T dS = dU + P dV = a d(VT^4) + \frac{a}{3} T^4 dV,$$

or

$$\frac{4}{3} T^4 dV + 4VT^3 dT = \frac{4}{3} T^4 d\left(\frac{4\pi}{3} r^3\right) + 4\left(\frac{4\pi}{3} r^3\right) T^3 dT = 0$$

or

$$Td\!r + rdT = 0,$$

which integrates to

$$T = T_{\text{dec}} \frac{r_{\text{dec}}}{r},$$

and since the radius of the comoving volume is proportional to the cosmic scale factor, we have

$$T = T_{\text{dec}} \frac{R_{\text{dec}}}{R}.$$

Today, we have $T_0 = 2.73 \, K$, so

$$\frac{R_0}{R_{\text{dec}}} \approx \frac{4000}{2.73} = 1465.$$

The cosmic scale factor has increased by a factor of about 1500 since decoupling. What we see as the CMBR today at any instant is the light emitted from a spherical surface. Since it is very nearly isotropic, the density on this (enormous) sphere must have been nearly uniform.

At the time of decoupling, the universe consisted of photons, mostly with frequencies near the optical range, and partially ionized hydrogen, and a few $\alpha$-particles. Electromagnetic and gravitational forces dominate. The universe has zero net charge, so electrical forces cancel out, on average. Gravity, on the other hand, adds up, so we must ask whether that uniform density state is stable when gravity is present. The answer must surely be no, since matter is not at all uniformly distributed today, instead clumping into galaxies, stars, planets, and so on. There must surely have been tiny fluctuations in the density (protogalaxies) at $t_{\text{dec}}$ that have since grown to be the galaxies that we see. What happens is that a tiny fluctuation is unstable. This instability may be seen as follows, in a nonrelativistic, static universe. Considering the massive particles as a fluid with a velocity $\vec{v}$, we have the Euler equation:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \nabla p + \vec{g},$$

(52)
the mass conservation equation
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \] (53)
and the field equation
\[ \nabla \cdot \vec{g} = -4\pi G\rho. \] (54)

Here \( \rho \) is the mass density, \( \vec{g} \) is the local gravitational acceleration, and \( p \) is the pressure. The uniform solution to these equations is \( \vec{v} = 0, \; \vec{g} = 0, \; \rho(\vec{x}, t) = \rho_0, \; p(\vec{x}, t) = p_0 \), and

We now wish to consider small perturbations \( \vec{v} = \delta \vec{v}, \; \rho = \rho_0 + \delta \rho, \) etc. This leads to
\[ \frac{\partial (\delta \vec{v})}{\partial t} = -\frac{v_s^2}{\rho_0} \nabla (\delta \rho) + \delta \vec{g}, \] (55)
\[ \frac{\partial (\delta \rho)}{\partial t} + \rho_0 \nabla \cdot (\delta \vec{v}) = 0, \] (56)
\[ \nabla \cdot \delta \vec{g} = -4\pi G(\delta \rho). \] (57)

Here \( v_s \) is the speed of sound, taken to be the adiabatic variation \( v_s = \delta p / \delta \rho \). Taking the divergence of the first equation, the derivative with respect to time of the second, and substituting from the third leads to
\[ \frac{\partial^2 (\delta \rho)}{\partial t^2} = v_s^2 \nabla^2 (\delta \rho) + 4\pi G\rho (\delta \rho). \] (58)

This wave equation has the usual solutions
\[ \delta \rho(\vec{x}, t) = \sum_k a_k e^{i\vec{k} \cdot \vec{x} - i\omega t}, \] (59)
where
\[ \omega^2 = v_s^2 k^2 - 4\pi G\rho. \] (60)

When \( k > k_J = \sqrt{4\pi G\rho / v_s} \), then \( \omega \) is real and this solution is just a sound wave that propagates (and dies away if we include dissipation). When \( k < k_J \), \( \omega \) is imaginary and the disturbance grows exponentially with an e-folding rate \( v_s (k_J^2 - k^2)^{1/2} \). The \( J \) memorializes Jeans, who first did this analysis.

This perturbative analysis only gives an idea of how the clump starts. Once it enters the nonperturbative regime and becomes a bound complex, its growth slows and eventually stops, becoming a galaxy, or a cluster of galaxies, or something. This regime is still an area of very active research.

These small density fluctuations in the universe at the time of decoupling are observable today as small variations in the temperature of the CMBR as a function of angle in the sky. The temperature of the light we see today
was determined by the local temperature of its environment at $t_{dec}$. The local temperature at that time is related to the local density by

$$\frac{\delta T}{T} = \kappa \frac{\delta \rho}{\rho},$$

(61)

where $\kappa$ is a number of order 1. For adiabatic perturbations, $\kappa = 1/3$.

The basic picture of the density fluctuations is this. Before the time of decoupling, the radiation, which exerts pressure on the matter but does not itself clump, keeps the density $\rho$ quite uniform. At $t_{dec}$ this situation changes, and any fluctuations can start to grow. (They actually do not grow exponentially as in the static universe model, but more slowly). By working backward from the density fluctuations we see today, we can estimate that at $t_{dec}$, $\delta \rho/\rho \sim \delta T/T \sim 10^{-4}$ to $10^{-5}$ and that the spatial scale of these fluctuations corresponds to angles less than 1 degree. Recent observations of these temperature fluctuations are in good agreement with detailed calculations of the density fluctuations at $t_{dec}$.