0.1 Cosmology

In order to study the thermodynamics of the cosmos, we will need to treat some additional topics in statistical mechanics. In addition, we need to review the relevant parts of particle physics and gravitation theory.

0.1.1 Chemical reactions and Mixtures

The very early universe was unimaginably hot, and the particles had very high energies, greater than their rest masses. Under these conditions, there is enough thermal energy to drive elementary particle reactions. This means that we must study the thermodynamics of reacting mixtures, a topic we have left aside until now. We will approach it using the language of chemistry. It turns out that thermal particle reactions follow exactly the same laws as chemical reactions.

We have so far limited our considerations to pure substances. In the grand canonical ensemble, the number of particles $N$ could change, and the driving force for changes in $N$ was the chemical potential $\mu$. If our system is a mixture of $I$ kinds of particles, the entropy $S_V$ of the reservoir will depend differently on the numbers $N_i$ of each, in general. So we define

$$\mu_i = -k_B T \left( \frac{\partial S_V}{\partial N_i} \right)_V$$

and we find the free energies

$$dA = -SdT - PdV + \sum_{i=1}^{I} \mu_i dN_i$$

$$dG = -SdT + VdP + \sum_{i=1}^{I} \mu_i dN_i$$

and so on: our earlier equations for one species are modified only by summing over the species of particles. From these equations, we have for particle 1 that

$$\mu_1 = \left( \frac{\partial A}{\partial N_1} \right)_{V,T,N_2,N_3,\ldots,N_I}$$

$$\mu_1 = \left( \frac{\partial G}{\partial N_1} \right)_{P,T,N_2,N_3,\ldots,N_I},$$

and similarly for the rest. Arguing by analogy to the way we argued for pure substances, we can easily establish that mixtures in diffusive contact with one another must have the same $\mu_i$ for all $i$ once they have come to equilibrium, and all of the $\mu_i$ must be spatially constant in any mixture that is in equilibrium.
Reactions  Now we are going to let the various components react. What this means is that there is a process that can convert one set of species into another set. The only constraint is that the number of atoms of each element must be conserved in the reaction. For example, we might consider a mixture of molecular hydrogen \( H_2 \), molecular oxygen \( O_2 \), and water \( H_2O \). These undergo a reaction

\[
2H_2 + O_2 \leftrightarrow 2H_2O. \tag{6}
\]

All reactions can go in both directions. Energy must be supplied to the endothermic direction

\[
2H_2 + O_2 \leftrightarrow 2H_2O, \tag{7}
\]

while energy is given off in the exothermic direction

\[
2H_2 + O_2 \rightarrow 2H_2O. \tag{8}
\]

The most general reaction can be written as

\[

\nu_A A + \nu_B B + ... \leftrightarrow -\nu_{A'} A' - \nu_{B'} B' + ...
\]

where the minus signs on the right-hand side are a convention. The \( \nu_i \) on the right side of the reaction equation are negative. For our example

\[
\nu_{H_2} = 2, \nu_{O_2} = 1, \nu_{H_2O} = -2. \tag{10}
\]

The values of the \( \nu_i \) can be seen as determining a constraint on the chemical potentials of the constituents, one that is crucial in the universe at early times. Consider a system held at constant temperature and pressure. Then the Gibbs free energy \( G \) is a minimum for all transformations that can occur. In particular, if a reaction takes place so that the concentrations change slightly, the Gibbs free energy is unchanged to first order. Let us look at this for the \( H_2 - O_2 - \) water system. A reaction \( 2H_2 + O_2 \rightarrow 2H_2O \) happens. We have

\[
\Delta G = -2 \left( \frac{\partial G}{\partial N_{H_2}} \right)_{P,T} - \left( \frac{\partial G}{\partial N_{O_2}} \right)_{P,T} + 2 \left( \frac{\partial G}{\partial N_{H_2O}} \right)_{P,T} = 0 \tag{11}
\]

but these derivatives are just the chemical potentials, so

\[
2\mu_{H_2} + \mu_{O_2} - 2\mu_{H_2O} = 0. \tag{12}
\]

This is an equation for the chemical potentials of the constituents once the reaction has reached equilibrium. The general result should now be clear. It is:

\[
\sum_i \nu_i \mu_i = 0. \tag{13}
\]

If we set up a system of reactants that are not initially in chemical equilibrium, the reaction will proceed until the chemical potentials of the reactants satisfy this linear condition.
0.1.2 Law of Mass Action

Let us see how this plays out in a very simple case, that of a mixture of ideal gases. Recall that for a single ideal gas we had a partition function

\[
Q = \frac{1}{h^3 N!} \int d^3 r_1 \ldots d^3 r_N \int d^3 p_1 \ldots d^3 p_N \, e^{-\beta \sum \epsilon_i / 2m} \left( e^{-\beta \sum \epsilon_i} \right)^N
\]

(14)

\[
= \frac{V^N}{N!} \left( \frac{m k_B T}{2 \pi \hbar^2} \right)^{3N/2}
\]

(15)

where \( \epsilon_i \) are the internal energy levels of the atoms or molecules. The free energy is

\[
A = -k_B T \ln Q = -N k_B T \ln (V/N) - N k_B T \ln q_{\text{int}}(T) - \frac{3}{2} N k_B T \ln \left( \frac{m k_B T}{2 \pi \hbar^2} \right)
\]

(16)

\[
= -N k_B T \ln (V/N) - N k_B T + N k_B f(T).
\]

(17)

Here I have defined the function \( f(T) \equiv -k_B T \ln q_{\text{int}}(T) - \frac{3}{2} k_B T \ln \left( \frac{m k_B T}{2 \pi \hbar^2} \right) \).

The chemical potential is

\[
\mu = \left( \frac{\partial A}{\partial N} \right)_V = -k_B T \ln (V/N) + f(T) = -k_B T \ln (k_B T / P) + f(T)
\]

(18)

\[
= k_B T \ln (P) - k_B T \ln (k_B T) + f(T) = k_B T \ln (P) + \chi(T)
\]

(19)

with \( \chi(T) \equiv f(T) - k_B T \ln (k_B T) \).

Now consider an ideal gas made up of several constituents. They do not interact, so the chemical potential of each constituent can be calculated separately:

\[
\mu_i = k_B T \ln (P_i) + \chi_i(T).
\]

(20)

The \( P_i \) are the partial pressures of the constituents:

\[
P = \sum_{i=1}^{l} P_i
\]

(21)

and the \( \chi_i \) may be different, since \( q_{\text{int}} \) may be different for different molecules.

Now the famous law of mass action can be deduced from Eq. 13. Using our result for the chemical potentials of the constituents of such a system, we have

\[
k_B T \sum_i \nu_i \ln P_i + \sum_i \nu_i \chi_i(T) = 0,
\]

(22)

which can be rewritten as

\[
\sum_i \ln [(P_i)^{\nu_i}] = -\beta \sum_i \nu_i \chi_i(T)
\]

(23)
or

$$\Pi_i ((P_i)^{\nu_i}) = \exp \left[ -\beta \sum_i \nu_i \chi_i(T) \right] = K_c(T). \quad (24)$$

The pressures exerted by the different coefficients are such that their product, properly weighted, is a function of the temperature alone. This may also be expressed in terms of the concentrations since $n_i = N_i / N = P_i / P$, which are also the fractional partial pressures:

$$\Pi_i ((n_i)^{\nu_i}) = \Pi_i ((P_i / P)^{\nu_i}) \quad (25)$$

$$= (1/P)^{\sum \nu_i} \exp \left[ -\beta \sum_i \nu_i \chi_i(T) \right] \quad (26)$$

$$= \Pi_i ((1/P)^{\nu_i}) K_c(T) \equiv K(P, T). \quad (27)$$

The function $K(P, T)$ is called the chemical equilibrium constant. Often enough, we are mainly worried about equilibrium at room temperature and atmospheric pressure, and, for a given set of reactants, $K$ is indeed a constant and can be given in reference books. Working our example, we find

$$\frac{(n_{H_2})^2 n_{O_2}}{(n_{H_2}O)^2} \ln K_c(T) = \left( \frac{1}{P} \right)^{\sum \nu_i} K_c(T) = \left( \frac{1}{P} \right)^{2+1-2} K_c(T) = \left( \frac{1}{P} \right) K_c(T). \quad (28)$$

You can see how the law of mass action can come about from a probabilistic argument. For a reaction $2H_2 + O_2 \leftrightarrow 2H_2O$ to take place in the forward direction, we must have 2 $H_2$ molecules and 1 $O_2$ molecule present in a reaction volume. The chance of having 2 $H_2$ molecules present in a given volume is proportional to $(n_{H_2})^2$, not $n_{H_2}$, and so on. So the rate of the reaction to the right is proportional to $(n_{H_2})^2 (n_{O_2})$. The rate of the reaction to the left is proportional to $(n_{H_2}O)^2$, since we need two water molecules to make an $O_2$. In equilibrium, the rate of the reaction to the left and to the right are equal, so $(n_{H_2})^2 n_{O_2}$ is proportional to $(n_{H_2}O)^2$.

It is of great practical importance that the law of mass action also holds in weak solutions. The usual situation is one where the solvent molecules vastly outnumber the solute molecules, but the solute molecules are the ones that are undergoing the reaction, and the solvent is inert. The reaction can proceed much more quickly because even a weak solution can contain a much higher density of reactants than a gas. This is one reason that chemistry is often done in solutions. The chemical potential of a given kind of solute molecule is related to its partial pressure just as in a gas. From this the law of mass action follows immediately.

The reaction rates have not been mentioned at all. This is because the rates are irrelevant for determining the equilibrium state. The presence or absence of catalysis is similarly irrelevant. In real life, chemical equilibrium is not always reached, and the concentration of a reactant may be determined by kinetics
instead of equilibrium conditions. This will be the case if the rate of any of the reactions is slow compared with the time the system has been in the constant external conditions.

0.2 Short review of elementary particle physics

We need only relatively few facts from particle physics.

The elementary particles are quarks and leptons, which are fermions, and the bosons that carry the forces between them.

Quarks carry color charge, which is the charge for the strong interactions that hold nuclei together. In the very early, extremely hot, universe, quarks must become free, in some sense. But this part of the universe’s history is not well understood, and we shall not enter into it. In the epoch that we shall treat, we will not need to know anything about the quarks and the strong force except that the quarks form bound states known as the proton \( p \) and neutron \( n \) as the two most stable (lowest energy) hadrons. These can further combine into nuclei such as the deuteron \( d = ^2 H = n + p \), the tritium nucleus \( ^3 H = p + 2n \), \( ^3 \text{He} = 2p + n \), and \( \alpha = ^4 \text{He} = 2n + 2p \).

Leptons are the charged spin 1/2 tau \( \tau^- \), muon \( \mu^- \), and electron \( e^- \), together with their antiparticles the \( \tau^+ \), \( \mu^+ \), and \( e^+ \). Each particle has spin degeneracy 2. There are also the partner neutrinos the \( \nu_\tau \), \( \nu_\mu \), and \( \nu_e \) together with their antiparticles the \( \bar{\nu}_\tau \), \( \bar{\nu}_\mu \), and \( \bar{\nu}_e \). The neutrinos also have spin 1/2, but exist only in one of the two states, the neutrinos being left-handed, and the antineutrinos being right-handed, a consequence of parity violation. Thus the spin degeneracy of each is 1.

Charged leptons and hadrons interact electromagnetically, via the exchange of photons \( \gamma \). Photons are massless bosons with spin 1, but come in only right-handed and left handed varieties - their spin degeneracy is 2.

All leptons and hadrons interact weakly with the exchange of \( W \) and \( Z \) bosons. The \( W \) and \( Z \) are massive bosons with spin 1 but are not stable and will not concern us here.

We will be concerned largely with reactions of various kinds. The reactions coming from the strong force that will be of interest are:

\[
\begin{align*}
  p + n & \leftrightarrow d + \gamma & (29) \\
  d + d & \leftrightarrow \ ^3\text{He} + n & \leftrightarrow ^3 H + p & (30) \\
  ^3 H + d & \leftrightarrow ^4\text{He} + n & (31) \\
  p + d & \leftrightarrow ^3\text{He} + \gamma & (32) \\
  n + d & \leftrightarrow ^3 H + \gamma & (33) \\
  p + ^3 H & \leftrightarrow ^4\text{He} + \gamma & (34) \\
  n + ^3 \text{He} & \leftrightarrow ^4\text{He} + \gamma & (35) \\
  d + d & \leftrightarrow ^4\text{He} + \gamma. & (36)
\end{align*}
\]
These processes are responsible for nucleosynthesis in the early universe. The reason for the appearance of $\gamma$-rays in these strong reactions (even though the $\gamma$ has no color charge) is that the nucleus may be created in an excited state that decays to the ground state by $\gamma$-emission.

Electromagnetic reactions are too numerous to list exhaustively. All charged particles can take part, and any number of photons can be created:

$$P^+ + P^- \leftrightarrow n\gamma.$$  

(37)

Here $n$ is an arbitrary integer and $P^\pm$ is any particle and its antiparticle. More generally, any set of charged particles can go into any other with the production of an arbitrary number of photons, subject only to the conservation of electric charge, baryon number, and electron-, muon-, and tau-type lepton numbers.

All hadrons and leptons can take part in the weak interactions. The most important of these for our purposes are $\beta$-decay of the neutron and variants of it:

$$n \leftrightarrow p + \ell^- + \bar{\nu}_\ell$$  \hspace{1cm} (38)

$$n + \nu_\ell \leftrightarrow p + \ell^-$$  \hspace{1cm} (39)

$$n + \ell^+ \leftrightarrow p + \bar{\nu}_\ell.$$  \hspace{1cm} (40)

The fact that these reactions take place by the exchange of the massive $W$ and $Z$ bosons makes no difference for our purposes, because these bosons are very unstable.

### 0.3 Short Review of General Relativity

Most students at this level have not studied the relativistic theory of gravity in detail, so I give a short discussion here. The theory in its most general form is quite complicated, but we will only be applying it to the universe as a whole and we will make a very strong simplifying assumption about the universe, namely that it satisfies the cosmological principle. Under this assumption, the pertinent parts of general relativity are fortunately quite simple.

#### 0.3.1 The cosmic scale factor

The cosmological principle says that the universe is homogeneous and isotropic on sufficiently large length scales. The evidence for this is, in my opinion, not entirely convincing. Even on the longest scales in galactic mapping, there still appears to be some spatial structure, possibly of a fractal kind. Still, the cosmological principle leads to observable consequences that are borne out to a large degree. In any case, it is the only hypothesis that is simple enough to be strongly tested by observation at present. It works well enough that whatever replaces it, such as fractality of some kind, is sure to be different from it in subtle and not gross ways. Let us think about the most general homogeneous
and isotropic space in two dimensions. A flat plane is one such space. A sphere, which has constant positive curvature, is another. It is also possible to make 2-surfaces that are homogeneous and isotropic but that have constant negative curvature, (like a saddle point). In 4-dimensional spacetime, we can make a 3-sphere that has constant curvature. This is of course not so easy to visualize.

Under the hypothesis of spatially constant curvature, one can show (although we shall not) that the proper distance $d\tau$ between two spacetime events whose coordinates differ by $(dt, dr, d\theta, d\phi)$ is given by the Robertson-Walker metric

$$c^2 d\tau^2 = c^2 dt^2 - R^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right),$$

and $k = 1$ for a closed universe with positive curvature, $k = 0$ for a flat universe (currently considered most likely), and $k = -1$ for an universe with negative curvature. The units for the dimensionless coordinate $r$ are chosen so that $k$ has the simple values given. For $k = 1$ the universe is finite in volume, and a three-dimensional slice at a constant time is a 3-sphere. $R(t)$ may be thought of the radius of this sphere when it is embedded in (an imaginary) 4-dimensional space. For $k = 0$ or $k = -1$ the universe is always infinite in volume, and the interpretation of $R(t)$ as a radius of something is untenable. We will just call it the cosmic scale factor. We shall also set $c = 1$ henceforth.

The Big Bang is not like an explosion. It does not start at some particular point in space and spread out. Instead, it just represents a point in time when $R = 0$. We define this as the zero of time.

The coordinates $(t, r, \theta, \phi)$ are defined so that a typical galaxy will have constant $r$, $\theta$, and $\phi$. These are therefore called “comoving” coordinates. (We may take the Milky Way or any other galaxy at the origin.) The physical meaning of these coordinates may be made more concrete by recalling that for any two points connected by a light ray $d\tau = 0$. So if the the crest of a light ray that leaves a distant galaxy at $r = r_G$ at $t = t_G$ along the line $\theta = \text{const.}$, $\phi = \text{const.}$ arrives at the Earth at time $t_0$ it follows the equation of motion

$$0 = dt^2 - R^2(t) \frac{dr^2}{1 - kr^2},$$

so

$$\int^{t_0}_{t_G} \frac{dt}{R(t)} = \int^{0}_{-r_G} \frac{dr}{\sqrt{1 - kr^2}}.$$

The next crest leaves at a distant galaxy local time $t_G + \tau$ later and arrives at the Earth at $t_0 + \tau$:

$$\int^{t_0 + \tau}_{t_G + \tau} \frac{dt}{R(t)} = \int^{0}_{-r_G} \frac{dr}{\sqrt{1 - kr^2}}.$$
Subtracting these two equations, we find
\[
\int_{t_G}^{t_0+\tau_s} \frac{dt}{R(t)} = \int_{t_G+\tau}^{t_0} \frac{dt}{R(t)}
\]  \hspace{1cm} (45)
and rearranging, we have
\[
\int_{t_G}^{t_G+\tau} \frac{dt}{R(t)} = \int_{t_0}^{t_0+\tau_s} \frac{dt}{R(t)}.
\]  \hspace{1cm} (46)
Since \(R(t)\) can be considered to be constant over the period of a light oscillation, we have
\[
\frac{\tau}{R(t_G)} = \frac{\tau_s}{R(t_0)}.
\]  \hspace{1cm} (47)
Here \(\tau\) is the period of the oscillation in its own frame and \(\tau_s\) is the red-shifted value. The red shift is expressed conventionally by
\[
z = \frac{\lambda_0}{\lambda} - 1 = \frac{\tau_s}{\tau} - 1 = \frac{R(t_0)}{R(t_G)} - 1,
\]  \hspace{1cm} (48)
so the function \(R(t)\) determines the redshift of light that we see from distant galaxies. More generally, \(R(t)\) is the quantity that determines motion in the cosmical gravitational field. If the universe is closed \((k = 1)\), it makes some sense to speak of \(R(t)\) as the "radius of the universe". If \(k = 0\) or \(k = -1\), then there is not much point in this phrase. \(R(t)\) is the "cosmic scale factor". Also note that the cosmological redshift is not really very similar to the gravitational redshift that happens when a photon is launched from the strong gravitational field of the sun and detected on earth. That shift is not symmetrical - it is blue in one direction and red in the other. The cosmological shift is red in both directions, more like a Doppler shift.

Notice that we have assumed that \(t\) is a local time, that is, it is measured by a standard clock at the distant galaxy in question, which makes \(t\) a sort of cosmic standard time. We will say a bit more later about how we can actually define this time operationally.

Because of the definition of the coordinates as "comoving", the density of galaxies \(n_G(t)\) satisfies
\[
n_G(t)R^3(t) = \text{const.},
\]  \hspace{1cm} (49)
which means that the distance between any two galaxies (whose random velocities are assumed to negligible) is in proportion to \(R(t)\). The observation of red shifts rather than blue shift for distant galaxies implies that \(dR/dt > 0\) at the present time.

Determining \(R(t)\) is one of the basic tasks of cosmology. We expand it as
\[
R(t) = R(t_0) = \left[1 + H_0(t-t_0) - \frac{1}{2}g_0H_0^2(t-t_0)^2 + ...\right]
\]  \hspace{1cm} (50)
where \( t_0 \) is the present time, and the Hubble constant is defined as

\[
H_0 = \frac{1}{R(t_0)} \frac{dR}{dt}(t_0).
\]  

(51)

The long controversy about the value of \( H_0 \) has finally settled down, and we now know that \( H_0 = 72 \pm 7 \text{ km/s/Mpc} \) in the units peculiar to this constant alone. \( 1 \text{ Mpc} = 3.1 \times 10^{19} \text{ km} \), so \( H_0 = 2.3 \times 10^{-18} / \text{s} \) and \( 1/H_0 = 1.4 \times 10^{10} \text{ yr} \) is a rough estimate of the age of the universe. \( q_0 \) is currently thought to be negative so that the expansion is accelerating.

### 0.3.2 The Einstein Equation

The Einstein equation is the analog of Maxwell’s equations for the electromagnetic field. It gives the relation between the source of the gravitational field, which is mass-energy, and the field itself. As we saw in the previous section, the cosmical gravitational field is summarized in the function \( R(t) \) and the constant \( k \). Einstein’s equation for \( R(t) \) is

\[
\left( \frac{dR(t)}{dt} \right)^2 + k = \frac{8\pi G}{3} \rho(t) R(t)^2,
\]  

(52)

where \( \rho \) is the mass-energy density of the universe. The cosmological principle implies that \( \rho \) does not depend on position.

In the Maxwell theory we also have the equation for current conservation. The analogous equation in gravitation theory is the energy conservation equation:

\[
\frac{d}{dR} (\rho R^3) = -3PR^2,
\]  

(53)

where \( P \) is the pressure (also position-independent). We can put this into a more familiar form if we write it as

\[
dU = \frac{4\pi}{3} (\rho R^3) - PdV = \frac{4\pi P}{3} d(R^3).
\]  

(54)

This is the first law of thermodynamics (for the entire universe!). Thus we can, for our purposes, actually think of the universe as an expanding balloon.

These two equations, together with an equation of state to relate \( P \) and \( \rho \), determine the evolution of the universe. We use the equation of state to eliminate \( \rho \) from 53. This determines \( \rho \) as a function of \( R \). This in turn allows us to solve 52 for \( R(t) \). In the present-day universe \( P << \rho \) because \( \rho \) comes mostly from matter moving nonrelativistically. (We are maybe not used to even thinking in these terms, but \( P \) does have the units of energy/volume. If you look back at our formulas for the pressure of a nonrelativistic gas of massive particles, you see that \( P = nkT << nmc^2 \), where \( n \) is the number density and
\( m \) is the mass of whatever particles make up the gas. On the other hand, if you recall the expression \( P = U/3 \) for the photon gas, you see that pressure cannot be neglected compared to energy density in that case.) We can neglect the RHS of Eq. 53, so \( \rho = \rho_0 R_0^3 / R^3 \). A 0 subscript indicates present-day values. If we take \( k = 0 \) then Eq. 52 becomes
\[
\left( \frac{dR}{dt} \right)^2 = \frac{8 \pi G \rho_0 R_0^3}{3R}
\tag{55}
\]
or
\[
\int_0^R R^{1/2} dR = \frac{2}{3} R^{3/2} = \left( \frac{8 \pi G \rho_0 R_0^3}{3} \right)^{1/2} \int_0^t dt = \left( \frac{8 \pi G \rho_0 R_0^3}{3} \right)^{1/2} t \tag{56}
\]
so
\[
R = \left( 4 \pi G \rho_0 R_0^3 \right)^{1/3} t^{2/3}.
\tag{57}
\]
While, as we shall see, this solution cannot hold to early times, it does show that the universe can have solutions in which \( R(t = 0) = 0 \). Also, the fact that \( R(t) \) is an increasing function with negative curvature \( (dR/dt > 0 \text{ and } d^2 R/dt^2 < 0) \) is a general feature of the equations as they stand, even with \( k \neq 0 \).

Now let us drop the requirement that \( k = 0 \) and see what we can say about its value. Actually, Eq. 52 makes the issue of whether the universe is closed or open or flat fairly simple. First define the time-dependent Hubble constant
\[
H = \frac{1}{R} \frac{dR}{dt}
\tag{58}
\]
Then write Eq. 52 as
\[
\frac{k}{H^2 R^2} = \frac{\rho}{\rho_c} - 1 = \Omega - 1,
\tag{59}
\]
where
\[
\rho_c = \frac{3H^2}{8\pi G}
\tag{60}
\]
is the critical density and
\[
\Omega = \frac{\rho}{\rho_c}
\tag{61}
\]
is the ratio of the density to the critical density. At the present time
\[
\rho_c = 9.5 \times 10^{-30} \text{ g/cm}^3.
\tag{62}
\]

\( H^2 R^2 \) is positive, so \( k \) is positive if \( \Omega > 1 \), negative if \( \Omega < 1 \), and (the presently preferred alternative, and a consequence of inflation) \( k = 0 \) if \( \Omega = 1 \).

Observations presently strongly favor \( d^2 R/dt^2 > 0 \), i.e., \( q_0 \) is negative and the expansion of the universe is accelerating. This is not possible with the Einstein equations in the form that we have them. In order to have acceleration, we
need to add an additional term to the equations proportional to the so-called 'cosmological constant', usually denoted by $\lambda$. This term does not spoil the Lorentz invariance of the theory. It introduces a repulsive force between masses that is proportional to their separation. There is a window of values for this constant such that it is large enough to explain the cosmic acceleration and small enough that it does not spoil the successes of the theory of gravity on smaller scales. From the point of view of quantum field theory, however, no reasonable calculation can produce a value in this window.