A NLO Calculation of pQCD:
Total Cross Section of $P\overline{P} \rightarrow W^{+} + X$

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Outline

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$\Rightarrow$ **Born Cross Section**

2. Factorization Theorem

$\Rightarrow$ **How to organize a NLO calculation of pQCD**

3. Feynman rules and Feynman diagrams

$\Rightarrow$ **"Cut diagram" notation**

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$\Rightarrow$ **Dimensional Regularization**

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Appendices:

A. γ**-matrices in** n **dimensions**

B. **Some integrals and "special functions"**

C. **Angular integrals in** n **dimensions**

D. **Two-particle phase space in** n **dimensions**

E. **Explicit Calculations**

(Typesetting: prepared by Qing-Hong Cao at MSU.)

A few references can be found in

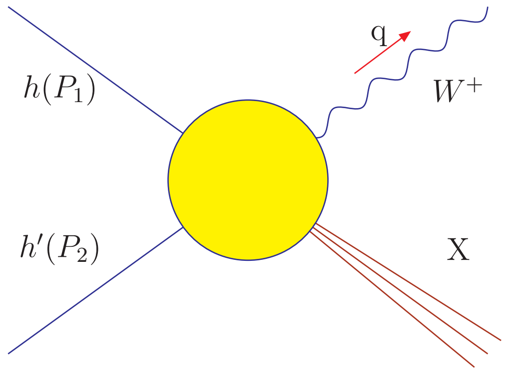
"Handbook of pQCD"

on CTEQ website

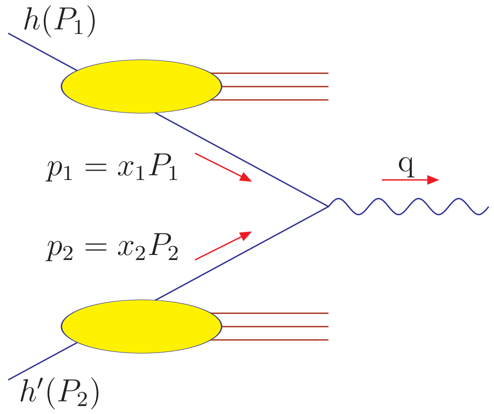
http://www.phys.psu.edu/∼ cteq/

Parton Model

$$\sigma_{hh'\to W^+X} =$$



$$=$$



$$\sigma_{hh'\to W^+X} = \sum_{f,f'=q,\bar{q}} \int_0^1 dx_1 dx_2 \left\{ \phi_{f/h}(x_1)\, \hat{\sigma}_{ff'} \phi_{\bar{f}'/h'}(x_2) + (x_1 \leftrightarrow x_2) \right\}$$

Partonic "Born"
Cross Section of $f\overline{f}' \to W^+$

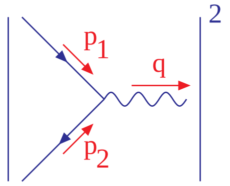
The probablility of finding a "parton" f with fraction x_1 of the hadron h momentum

Born Cross Section

$$\hat{\sigma}_{q\bar{q}'} = \frac{1}{2\hat{s}} \int \frac{d^3q}{(2\pi)^3\, 2q_0} (2\pi)^4\, \delta^4\left(p_1 + p_2 - q\right) \cdot \overline{|\mathcal{M}|^2}$$

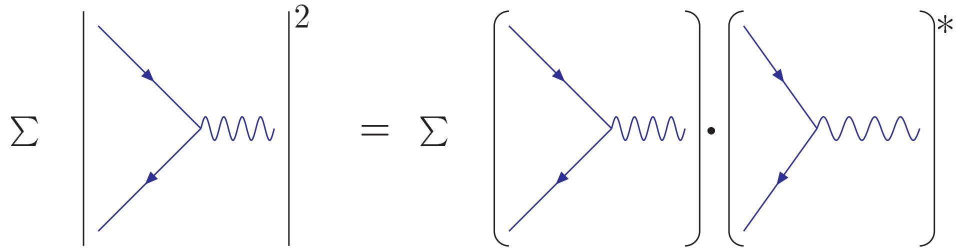
where

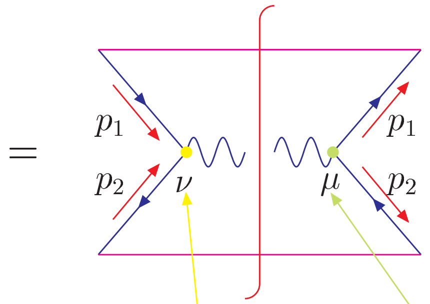
$$\overline{|\mathcal{M}|^2} = \underbrace{\left(\frac{1}{3}\cdot\frac{1}{3}\right)\left(\frac{1}{2}\cdot\frac{1}{2}\right)}_{\text{average color and spin}} \sum_{\substack{spin\\ color}}$$



$$\left[\ \text{Or,}\ \ -i\mathcal{M} = \bar{v}\left(p_2\right) \frac{ig_w}{\sqrt{2}} \gamma_\mu \frac{1}{2}\left(1-\gamma_5\right) u\left(p_1\right)\ \right]$$

"Cut-diagram" notation





$\frac{ig_w}{\sqrt{2}}\gamma_\nu P_L$

$-\frac{ig_w}{\sqrt{2}}\gamma_\mu P_L$

$P_L \equiv \frac{1}{2}(1-\gamma_5)$

$$= \left(\frac{g_W}{\sqrt{2}}\right)^2 Tr\left[\not{p}_1 \gamma_\mu P_L\ \not{p}_2 \gamma_\nu P_L\right] \cdot \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{M^2}\right) \cdot Tr\ I_{3\times 3}$$

Doesn't contribute for $m_q = 0$,
due to Ward identity

Color

$$\begin{aligned} & Tr\left[\not{p}_1\gamma_\mu P_L\ \not{p}_2\gamma^\mu P_L\right](-1) \\ = & Tr\left[\not{p}_1\gamma_\mu\ \not{p}_2\gamma^\mu P_L\right](-1) \\ = & (-2)\, Tr\left[\not{p}_1\ \not{p}_2 P_L\right](-1) \\ = & (-2)\cdot\frac{1}{2}\cdot 4\,(p_1\cdot p_2)\,(-1) \\ = & +2\hat{s} \end{aligned}$$

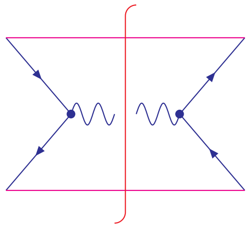
$$P_L P_L = P_L = \frac{1}{2}(1-\gamma_5)$$

$$\gamma_\mu\ \not{p}_2\gamma^\mu = -2\ \not{p}_2$$

$$Tr\left(\not{p}_1\ \not{p}_2\right) = 4\,(p_1\cdot p_2)$$

$$Tr\left(\not{p}_1\ \not{p}_2\gamma_5\right) = 0$$

$Tr\left[I_{3\times 3}\right] = 3$ $\left(\hat{s}\equiv (p_1+p_2)^2 = q^2 \text{ and } p_1^2 = p_2^2 = 0\right)$

$\Rightarrow$  $= \left(\frac{g_w}{\sqrt{2}}\right)^2\cdot(+2\hat{s})\,(3) = 3\ g_w^2\ \hat{s}$

$$\int\frac{d^3q}{2q_0}\delta^4(p_1+p_2-q) = \int d^4q\,\delta^4(p_1+p_2-q)\,\delta^+\left(q^2-M^2\right)$$
$$= \delta\left(q^2-M^2\right)$$

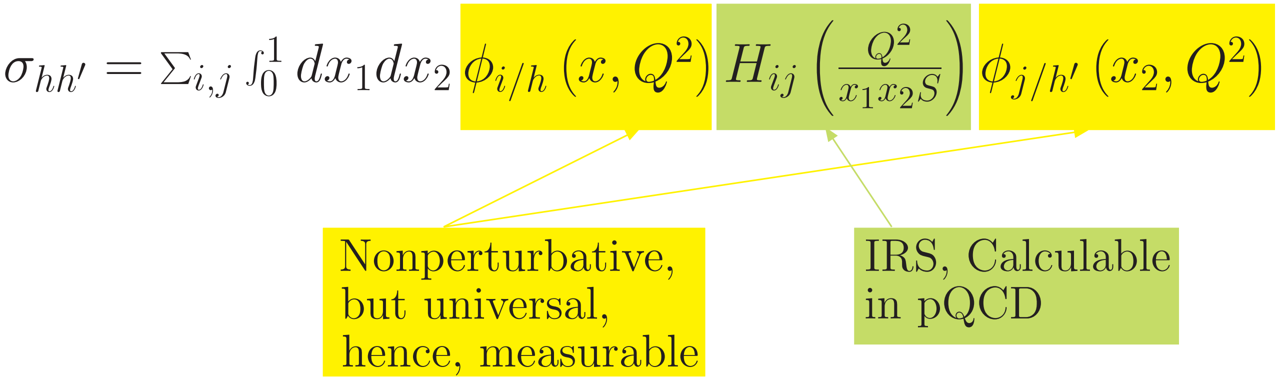
where M is the mass of W-boson.

Thus,

$$\begin{aligned} \hat{\sigma}_{q\bar{q}'} &= \frac{1}{2\hat{s}}(2\pi)\cdot\delta\left(\hat{s}-M^2\right)\cdot\left(\frac{1}{3}\right)\left(\frac{1}{2}\cdot\frac{1}{2}\right)\cdot g_w^2\hat{s} \\ &= \frac{\pi}{12}g_w^2\delta\left(\hat{s}-M^2\right) \\ &= \frac{\pi}{12\hat{s}}g_w^2\delta\left(1-\hat{\tau}\right) \end{aligned}$$

$$\left(\begin{array}{c} \hat{\tau} = M^2/\hat{s}\ ,\ \hat{s} = x_1x_2S \ \text{ for} \\ S = (P_1+P_2)^2 \ \text{ and } P_1^2 = P_2^2 = 0 \end{array}\right)$$

Factorization Theorem



$$\sigma_{hh'} = \Sigma_{i,j} \int_0^1 dx_1 dx_2 \, \phi_{i/h}\left(x, Q^2\right) H_{ij}\left(\frac{Q^2}{x_1 x_2 S}\right) \phi_{j/h'}\left(x_2, Q^2\right)$$

Nonperturbative,
but universal,
hence, measurable

IRS, Calculable
in pQCD

Procedure:

(1) Compute σ_{kl} in pQCD with k, l partons

(not h, h' hadron)

$$\sigma_{kl} = \sum_{i,j} \int_0^1 dx_1 dx_2 \, \phi_{i/k}\left(x_1, Q^2\right) H_{ij}\left(\frac{Q^2}{x_1 x_2 S}\right) \phi_{j/l}\left(x_2, Q^2\right)$$

(2) Compute $\phi_{i/k}, \phi_{j/l}$ in pQCD

(3) Extract H_{ij} in pQCD

H_{ij} IRS $\Rightarrow$ H_{ij} indepent of k, l

$\Rightarrow$ same H_{ij} with $(k \rightarrow h, l \rightarrow h')$

(4) Use H_{ij} in the above equation with $\phi_{i/h}, \phi_{j/h'}$

Extracting H_{ij} in pQCD

- Expansions in α_s:

$$\sigma_{kl} = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \sigma_{kl}^{(n)} \qquad\qquad \alpha_s = \frac{g^2}{4\pi}$$

$$H_{ij} = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n H_{ij}^{(n)}$$

$$\phi_{i/k}(x) = \delta_{ik}\delta(1-x) + \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \phi_{i/k}^{(n)}$$

$$\uparrow$$

$\phi_{i/k}^{(0)}$ ($\alpha_s = 0 \Rightarrow$ Parton k " stays itself ")

- Consequences:

$$H_{ij}^{(0)} = \sigma_{ij}^{(0)} = \text{“Born”}$$

suppress "^" from now on

$$H_{ij}^{(1)} = \boxed{\sigma_{ij}^{(1)}} - \left[\, \boxed{\sigma_{il}^{(0)}}\, \boxed{\phi_{l/j}^{(1)}} + \boxed{\phi_{k/i}^{(1)}}\, \sigma_{kj}^{(0)} \,\right]$$

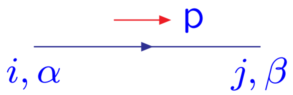
Computed from Feynman diagrams
(process dependent)

Computed from the definition of perturbative parton distribution function
(process independent, scheme dependent)

Feynman Rules

- Quark Propagator

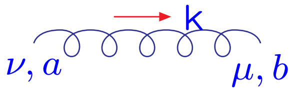
Take m=0 in
our calculation



$\frac{i(\not{p}+m)_{\beta\alpha}}{p^2-m^2+i\epsilon}\delta_{ij}$

(i,j=1,2,3)

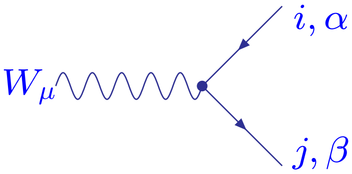
- Gluon Propagator



$\frac{i(-g_{\mu\nu})}{k^2+i\epsilon}\delta_{ab}$

(a,b=1,2...,8)

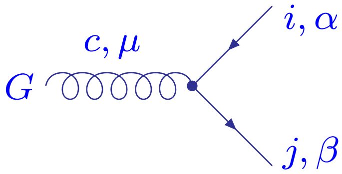
- Quark-W Vertex



$i\frac{g_W}{\sqrt{2}}(\gamma_\mu)_{\beta\alpha}\frac{(1-\gamma_5)}{2}\delta_{ij}$

$g_w = \frac{e}{\sin\theta_w}$, weak coupling

- Quark-Gluon Vertex



$-ig\,(t_c)_{ji}\,(\gamma_\mu)_{\beta\alpha}$

t_c is the $SU(N)_{N\times N}$ generator

- Quark Color Generators

$$[t_a, t_b] = i f_{abc} t_c$$

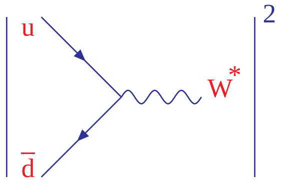
$$\sum_c t_c^2 = C_F\, I_{N\times N}$$

$$Tr(\sum_c t_c^2) = N\, C_F$$

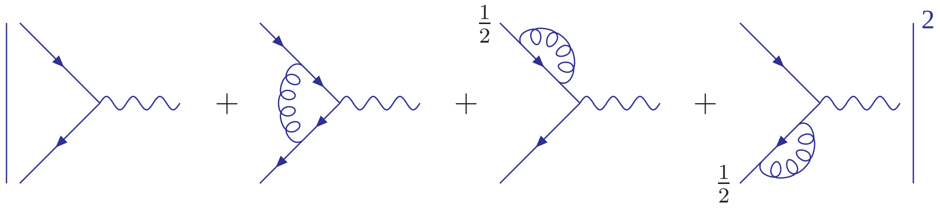
$$C_F = \frac{N^2-1}{2N} = \frac{4}{3}\ ,\ (N=3)$$

Feynman Diagrams

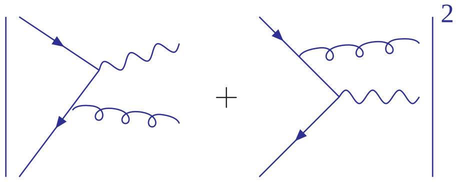
- Born level $\alpha_s^{(0)}$ $(q\overline{q'})_{Born}$



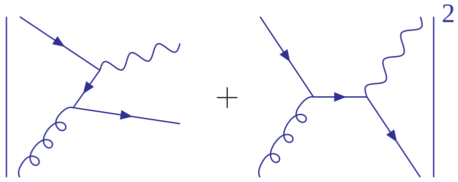
- NLO: $(\alpha_s^{(1)})$ virtual corrections $(q\overline{q'})_{virt}$



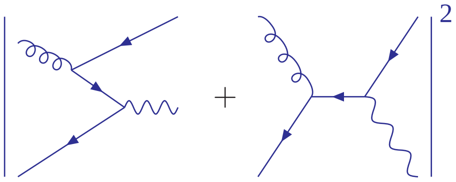
- NLO: $(\alpha_s^{(1)})$ real emission diagrams $(q\overline{q'})_{real}$



- NLO: $(\alpha_s^{(1)})$ real emission diagrams $(qG)_{real}$

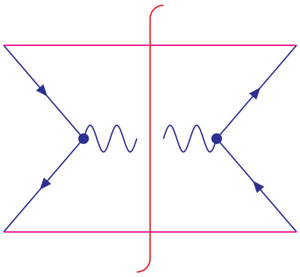


- NLO: $(\alpha_s^{(1)})$ real emission diagrams $(G\overline{q'})_{real}$

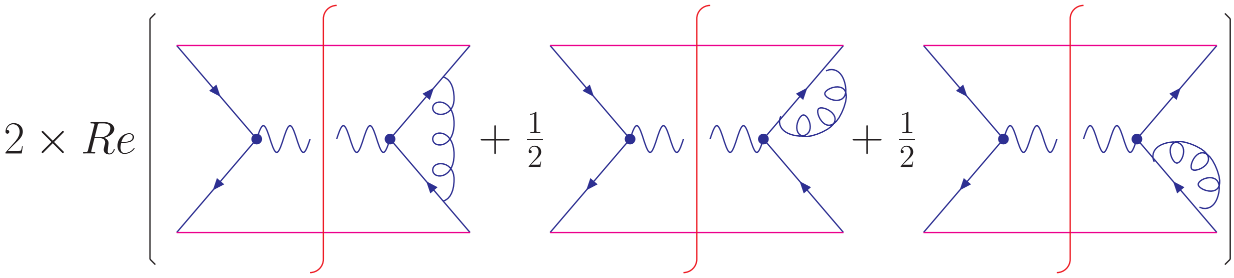


In "Cut-diagram" notation

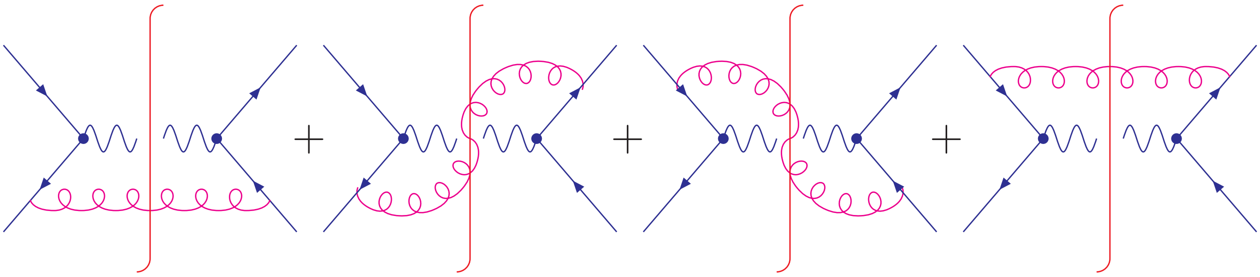
- $(q\overline{q'})_{Born}$



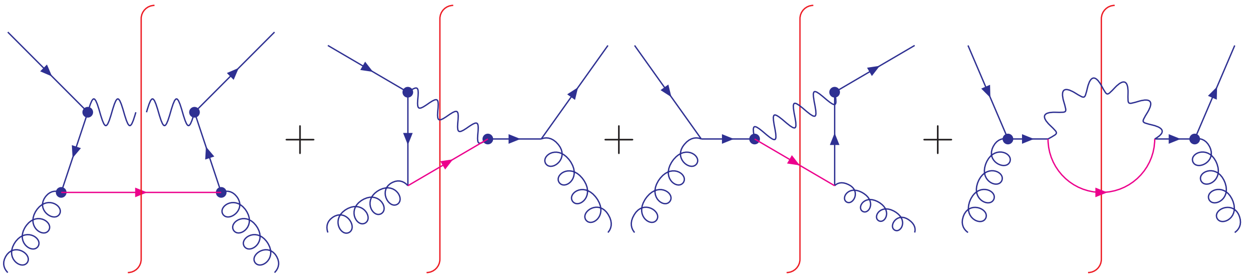
- $(q\overline{q'})_{virt}$



- $(q\overline{q'})_{real}$



- $(qG)_{real}$

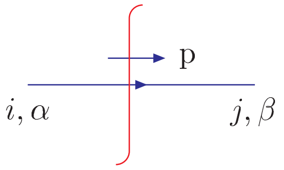


- $(G\overline{q'})_{real}$

Same as $(qG)_{real}$ after replacing q by $\overline{q'}$.

Feynman rules for cut-diagrams

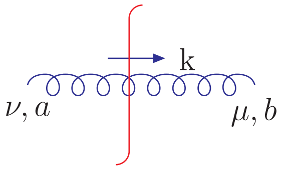
- quark line



$$(2\pi)\delta^+(p^2-m^2)(\not p+m)_{\beta\alpha}\delta_{ij}$$

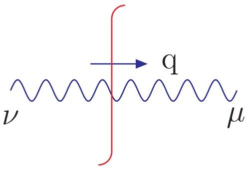
$\delta(p^2-m^2)\theta(p_0)$

- gluon line



$$(2\pi)\delta^+(k^2)(-g_{\mu\nu})\delta_{ab}$$

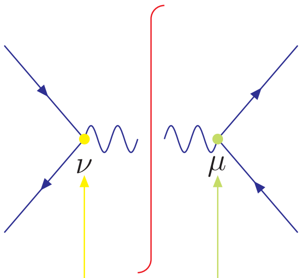
- W-boson line



$$(2\pi)\delta^+(q^2-M^2)(-g_{\mu\nu}+\frac{q_\mu q_\nu}{M^2})$$

Doesn't contribute for $m_f = 0$
because of Ward identity

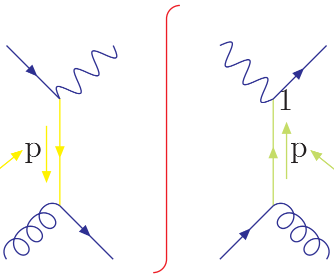
-



$i\frac{g_W}{\sqrt{2}}\gamma_\nu\frac{1}{2}(1-\gamma_5)$

$-i\frac{g_W}{\sqrt{2}}\gamma_\mu\frac{1}{2}(1-\gamma_5)$

-

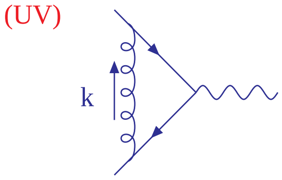


$\frac{i(\not p+m)}{p^2-m^2+i\epsilon}$

$\frac{-i(\not p+m)}{p^2-m^2-i\epsilon}$

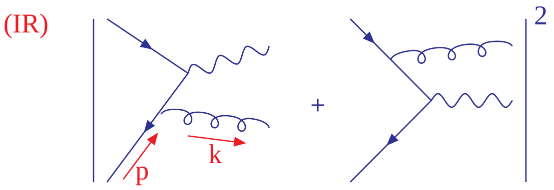
Immediate problems (Singularities)

- Ultraviolet singularity



$$\sim \quad \int^{\infty} d^4k \frac{\not{k}\ \not{k}}{(k^2)\,(k^2)\,(k^2)} \quad \to \quad \infty$$

- Infrared singularities



$$\to \quad \infty$$

as $k^\mu \to 0$ (soft divergence)

or $k^\mu \parallel p^\mu$ (collinear divergence)

$$\frac{1}{(p-k)^2 - m^2} = \frac{1}{-2p\cdot k} \quad \text{(for } m = 0 \text{ or } \ m \neq 0\text{)}$$

$p \cdot k \to 0$ as

$k \to 0$ or $k^\mu \parallel p^\mu$ (for $m = 0$)

$k \to 0$ (for $m \neq 0$)

(Similar singularities also exist in virtual diagrams.)

- Solutions

Compute H_{ij} in pQCD in $n = 4 - 2\varepsilon$ dimensions
(dimensional regularization)

(1) $n \neq 4 \Rightarrow$ UV & IR divergences appear as $\frac{1}{\varepsilon}$ poles
in $\sigma_{ij}^{(1)}$ (Feynman diagram calculation)

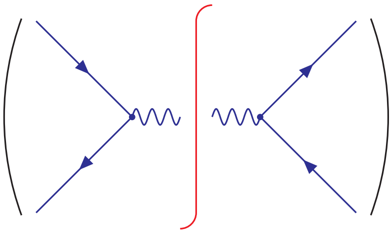
(2) H_{ij} is IR safe $\Rightarrow$ no $\frac{1}{\varepsilon}$ in H_{ij}
(H_{ij} is UV safe after "renormalization".)

Dimensional Regularization

(Revisit the Born Cross Section in n dimensions)

- $$\hat{\sigma}^{(0)}_{q\bar{q}'} = \frac{1}{2\hat{s}} \int \frac{d^{n-1}q}{(2\pi)^{n-1}\, 2q_0} (2\pi)^n \cdot \delta^n \left(p_1 + p_2 - q\right) \cdot \overline{|m|^2}$$

- $$\overline{|m|^2} = \left(\frac{1}{3} \cdot \frac{1}{3}\right) \left(\frac{1}{2} \cdot \frac{1}{2}\right) \cdot$$



In n-dim, the polarization degree of freedom is (2) for a quark, and (n-2) for a gluon.

- Using the Naive-γ^5 prescription:

$$Tr \left[\not{p}_1 \gamma_\mu P_L \ \not{p}_2 \gamma^\mu P_L \right] (-1)$$

$$= Tr \left[\not{p}_1 \gamma_\mu \ \not{p}_2 \gamma^\mu P_L \right] (-1) \qquad \gamma_\mu \ \not{p}_2 \gamma^\mu = -2\,(1-\varepsilon) \ \not{p}_2$$

$$= (-2)\,(1-\varepsilon)\, Tr \left[\not{p}_1 \ \not{p}_2 P_L \right] (-1)$$

$$= (-2)\,(1-\varepsilon) \cdot \frac{1}{2} \cdot 4\,(p_1 \cdot p_2)\,(-1)$$

$$= 2\,(1-\varepsilon)\,\hat{s}$$

- In n dimensions

$$\hat{\sigma}^{(0)}_{q\bar{q}'} = \frac{\pi}{12\hat{s}} g_w^2 \cdot (1-\varepsilon) \cdot \delta\,(1-\hat{\tau}) \equiv \sigma^{(0)} \cdot \delta(1-\hat{\tau})$$

Strong Coupling g in n dimensions

- In n dimensions

$$\int d^n x \mathcal{L}$$

$$\longrightarrow \int d^n x \left\{ \bar{\psi} i \not{\partial} \psi - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} + g t^a \bar{\psi} \gamma^\mu G_\mu \psi + \cdots \right\}$$

The dimension in mass unit (μ)

$$[\psi] \sim \mu^{\frac{n-1}{2}}$$

$$[G] \sim \mu^{\frac{n-2}{2}}$$

$$\left[\bar{\psi} G \psi\right] \sim \mu^{\frac{n-1}{2} \times 2 + \frac{n-2}{2}} = \mu^{\frac{3n}{2} - 2}$$

Since $\left[g \bar{\psi} G \psi\right] \sim \mu^n$, so

$$[g] \sim \mu^{\frac{-n}{2} + 2} \qquad n = 4 - 2\varepsilon$$

$$= \mu^{\varepsilon}$$

$\Rightarrow$ g has a dimension in mass when $\varepsilon \neq 0$

$\Rightarrow$ Feynman rules should read $g \to g\mu^{\varepsilon}$

Calculations

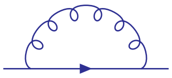
- Tools needed for a NLO calculation are collected in Appendices A-D

- The detailed calculation for each subprocess can be found in Appendices E

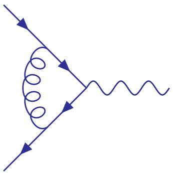
- In the following, I shall summarize the result for each subprocess

Virtual Corrections $(q\overline{q}')_{virt}$

(in Feynman Gauge)

-  $= \ 0$

  $\frac{1}{\varepsilon_{IR}}$and $\frac{1}{\varepsilon_{UV}}$ poles cancel when $\varepsilon_{UV} = -\varepsilon_{IR} \equiv \varepsilon$

- 

  $\frac{1}{\varepsilon_{UV}}$ cancel $\Rightarrow$ Electroweak coupling is not renormalized by QCD interactions at one-loop order (Ward identity, a renormalizable theory)

  $\frac{1}{\varepsilon_{IR}}$ poles remain

- $\sigma^{(1)}_{virt}$ is free of ultraviolent singularity.

$$\sigma^{(1)}_{virt} = \sigma^{(0)} \frac{\alpha_s}{2\pi} \delta\left(1 - \widehat{\tau}\right) \left(\frac{4\pi\mu^2}{M^2}\right)^{\varepsilon} \frac{\Gamma\left(1-\varepsilon\right)}{\Gamma\left(1-2\varepsilon\right)} \cdot \left\{ -\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \frac{2\pi^2}{3} \right\} \cdot \left(C_F\right)$$

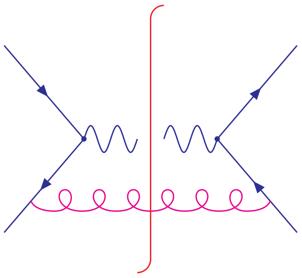
$-\frac{2}{\varepsilon^2}$: soft and collinear singularities

$-\frac{3}{\varepsilon}$: soft or collinear singularities

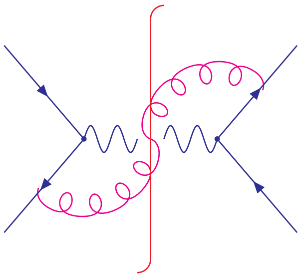
C_F: color factor

$\sigma^{(0)} \equiv \frac{\pi}{12\widehat{s}} g_w^2 \cdot (1 - \varepsilon)$

Real Emission Contribution $(q\bar{q}')_{real}$

- 

  $\sim \frac{1}{\varepsilon}$ **Collinear**

  $\sim \frac{1}{\varepsilon^2}$ **Soft and Collinear**

-

$$\sigma^{(1)}_{\text{real}}\left(q\bar{q}'\right) = \sigma^{(0)}\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{M^2}\right)^{\varepsilon}\frac{\Gamma\left(1-\varepsilon\right)}{\Gamma\left(1-2\varepsilon\right)}\cdot C_F$$

$$\cdot\left\{\frac{2}{\varepsilon^2}\delta\left(1-\widehat{\tau}\right) - \frac{2}{\varepsilon}\frac{1+\widehat{\tau}^2}{\left(1-\widehat{\tau}\right)_+} + 4\left(1+\widehat{\tau}^2\right)\left(\frac{\ln\left(1-\widehat{\tau}\right)}{1-\widehat{\tau}}\right)_+ - 2\frac{1+\widehat{\tau}^2}{1-\widehat{\tau}}\ln\widehat{\tau}\right\}$$

Note: $[\cdots]_+$ is a distribution,

$$\int_0^1 dz\ f\left(z\right)\left[\frac{1}{1-z}\right]_+ = \int_0^1 dz\frac{f\left(z\right)-f\left(1\right)}{1-z},$$ which is finite.

- In the soft limit, $\widehat{\tau}\longrightarrow 1 \quad \left(\widehat{\tau}=\frac{M^2}{\hat{s}}\right)$,

$$\sigma^{(1)}_{\text{real}}\left(q\bar{q}'\right) \longrightarrow \sigma^{(0)}\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{M^2}\right)^{\varepsilon}\frac{\Gamma\left(1-\varepsilon\right)}{\Gamma\left(1-2\varepsilon\right)}\cdot C_F$$

$$\cdot\left\{\frac{2}{\varepsilon^2}\delta\left(1-\widehat{\tau}\right) - \frac{4}{\varepsilon}\frac{1}{\left(1-\widehat{\tau}\right)_+} + 8\left(\frac{\ln\left(1-\widehat{\tau}\right)}{1-\widehat{\tau}}\right)_+\right\}$$

$\left(q\bar{q}'\right)_{virt} + \left(q\bar{q}'\right)_{\text{real}}$ at NLO

- $$\begin{aligned}\sigma^{(1)}_{q\bar{q}'} &= \sigma^{(1)}_{virt}\left(q\bar{q}'\right) + \sigma^{(1)}_{\text{real}}\left(q\bar{q}'\right)\\ &= \sigma^{(0)}\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{M^2}\right)^{\varepsilon}\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\cdot C_F\\ &\quad\cdot\left\{\frac{-2}{\varepsilon}\left(\frac{1+\hat{\tau}^2}{1-\hat{\tau}}\right)_+ - 2\frac{1+\hat{\tau}^2}{1-\hat{\tau}}\ln\hat{\tau} + 4\left(1+\hat{\tau}^2\right)\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+\right.\\ &\quad\left. + \left(\frac{2\pi^2}{3} - 8\right)\delta(1-\hat{\tau})\right\}\end{aligned}$$

Where we have used

$$\frac{-2}{\varepsilon}\left[\frac{1+\hat{\tau}^2}{(1-\hat{\tau})_+} + \frac{3}{2}\delta(1-\hat{\tau})\right] = \frac{-2}{\varepsilon}\left(\frac{1+\hat{\tau}^2}{1-\hat{\tau}}\right)_+$$

- All the soft singularities $\left(\frac{1}{\varepsilon^2}, \frac{1}{\varepsilon}\right)$ cancel in $\sigma^{(1)}_{q\bar{q}'}$

  $\Rightarrow KLN$ theorem

  (Kinoshita-Lee-Navenberg)

- $$\sigma^{(1)}_{q\bar{q}'} \sim \underset{\uparrow}{\frac{1}{\varepsilon}}(\text{term}) + \text{finite (terms)}$$

  Collinear Singularity

Factorization Theorem

- Perturbative PDF

$\phi^{(0)}_{i/k} = \delta_{ik}\delta\left(1 - x\right)$

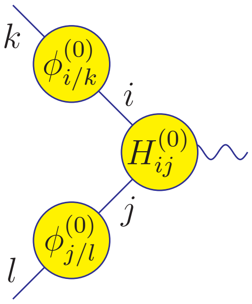
$\frac{\alpha_s}{\pi}\phi^{(1)}_{i/k}$ can be calculated from the definition of PDF.

(Process independent,but factorization scheme dependent)

-

(1)

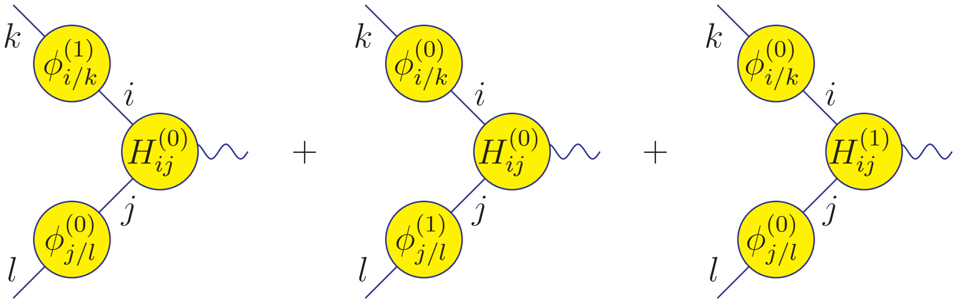
$\sigma^{(0)}_{kl} =$

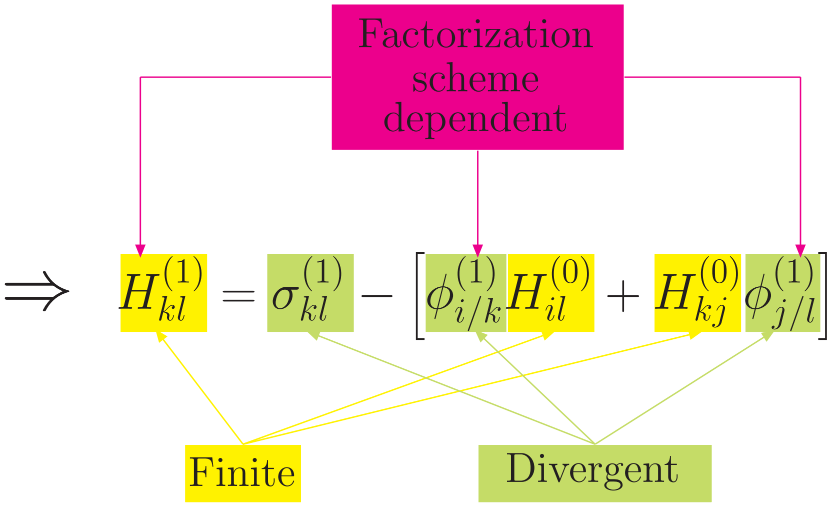


$\Rightarrow H^{(0)}_{kl} = \sigma^{(0)}_{kl}$

(2)

$\sigma^{(1)}_{kl} =$





Factorization scheme dependent

$$\Rightarrow H^{(1)}_{kl} = \sigma^{(1)}_{kl} - \left[\phi^{(1)}_{i/k} H^{(0)}_{il} + H^{(0)}_{kj}\phi^{(1)}_{j/l}\right]$$

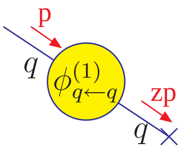
Finite

Divergent

Perturbative PDF

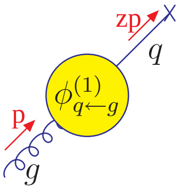
- In $\overline{MS}$-scheme (modified minimal subtraction)

$$\phi^{(1)}_{q/q}(z) = \phi^{(1)}_{\bar{q}/\bar{q}}(z) = \frac{-1}{\varepsilon}\frac{1}{2}\left(4\pi e^{-\gamma_E}\right)^{\varepsilon} P^{(1)}_{q\leftarrow q}(z)$$
$$\phi^{(1)}_{q/g}(z) = \phi^{(1)}_{\bar{q}/g}(z) = \frac{-1}{\varepsilon}\frac{1}{2}\left(4\pi e^{-\gamma_E}\right)^{\varepsilon} P^{(1)}_{q\leftarrow g}(z)$$

where the splitting kernel for 

is

$$P^{(1)}_{q\leftarrow q}(z) = C_F\left(\frac{1+z^2}{1-z}\right)_+$$
$$= C_F\left(\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right),$$

and for 

is

$$P^{(1)}_{q\leftarrow g}(z) = T_R\left(z^2 + (1-z)^2\right),$$

where $C_F = \frac{4}{3}$ and $T_R = \frac{1}{2}$.

$$\left(\begin{array}{l} \text{Note:The Pole part in the } \overline{MS} \text{ scheme is} \\ \qquad\qquad \frac{1}{\hat{\varepsilon}} = \frac{1}{\varepsilon}(4\pi e^{-\gamma_E})^{\varepsilon} = \frac{1}{\varepsilon} + \ln 4\pi - \gamma_E \\ \qquad \text{In the MS scheme, the pole part is just } \frac{1}{\varepsilon} \end{array}\right)$$

Find $H^{(1)}_{q\bar{q}'}$ (in the $\overline{MS}$ scheme)

- Take off the factor $\left(\frac{\alpha_s}{\pi}\right)$

$$\sigma^{(1)}_{q\bar{q}'} = \sigma^{(0)} \left\{ P^{(1)}_{q\leftarrow q}\left(\hat{\tau}\right) \left[\ln\left(\frac{M^2}{\mu^2}\right) - \frac{1}{\varepsilon} + \gamma_E - \ln 4\pi \right] \right.$$
$$\left. + C_F \left[-\frac{1+\hat{\tau}^2}{1-\hat{\tau}} \ln \hat{\tau} + 2\left(1+\tau^2\right) \left(\frac{\ln\left(1-\hat{\tau}\right)}{1-\hat{\tau}}\right)_+ + \left(\frac{\pi^2}{3} - 4\right) \delta\left(1-\hat{\tau}\right) \right] \right\}$$

-

$$H^{(1)}_{q\bar{q}'}\left(\hat{\tau}\right) = \sigma^{(1)}_{q\bar{q}'} - \left[2\phi^{(1)}_{q\leftarrow q}\sigma^{(0)}_{q\bar{q}'}\right]$$
$$= \hat{\sigma}^{(0)} \cdot \left\{ P^{(1)}_{q\leftarrow q}\left(\hat{\tau}\right) \ln\left(\frac{M^2}{\mu^2}\right) \right.$$
$$\left. + C_F \left[-\frac{1+\hat{\tau}^2}{1-\hat{\tau}} \ln \hat{\tau} + 2\left(1+\tau^2\right) \left(\frac{\ln\left(1-\hat{\tau}\right)}{1-\hat{\tau}}\right)_+ + \left(\frac{\pi^2}{3} - 4\right) \delta\left(1-\hat{\tau}\right) \right] \right\}$$

where

$$\hat{\tau} = \frac{M^2}{\hat{s}} = \frac{M^2}{x_1 x_2 S}, \qquad \sigma^{(0)} = \hat{\sigma}^{(0)} \cdot \left(1-\varepsilon\right),$$
$$\hat{\sigma}^{(0)} = \frac{\pi}{12\hat{s}} g_w^2 = \frac{\pi g_w^2}{12S} \frac{1}{x_1 x_2}.$$

- pQCD prediction

$$\sigma_{hh'} = \left\{ \sum_{f=q,\bar{q}'} \int dx_1 dx_2 \phi_{f/h}\left(x_1,\mu^2\right) \left[\sigma^{(0)}\delta\left(1-\hat{\tau}\right)\right] \phi_{\bar{f}/h'}\left(x_2,\mu^2\right) \right.$$
$$+ \sum_{f=q,\bar{q}'} \int dx_1 dx_2 \phi_{f/h}\left(x_1,\mu^2\right) \left[\frac{\alpha_s\left(\mu^2\right)}{\pi} H^{(1)}_{f\bar{f}}\left(\hat{\tau}\right)\right] \phi_{\bar{f}/h'}\left(x_2,\mu^2\right)$$
$$\left. + \sum_{f=q,\bar{q}'} \int dx_1 dx_2 \phi_{f/h}\left(x_1,\mu^2\right) \left[\frac{\alpha_s\left(\mu^2\right)}{\pi} H^{(1)}_{fG}\left(\hat{\tau}\right)\right] \phi_{G/h'}\left(x_2,\mu^2\right) + \left(x_1 \leftrightarrow x_2\right) \right\}$$

Find $H_{qG}^{(1)}$ (in the $\overline{MS}$ scheme)

- Take off the factor $\left(\frac{\alpha_s}{\pi}\right)$

$$\sigma_{qG}^{(1)} = \sigma^{(0)} \cdot \frac{1}{4} \cdot \left\{ 2P_{q\leftarrow g}^{(1)}(\hat{\tau}) \left[\frac{-1}{\varepsilon} \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} + \ln \frac{M^2(1-\hat{\tau})^2}{4\pi\mu^2\hat{\tau}} \right] + \frac{1}{2} + 3\hat{\tau} - \frac{7}{2}\hat{\tau}^2 \right\}$$

-

$$\begin{aligned} H_{qG}^{(1)}(\hat{\tau}) &= \sigma_{qG}^{(1)} - \left[\sigma_{q\bar{q}'}^{(0)} \phi_{\bar{q}'\leftarrow G}^{(1)}\right] \\ &= \frac{\hat{\sigma}^{(0)}}{2} \cdot \left\{ P_{q\leftarrow g}^{(1)}(\hat{\tau}) \left[\ln\left(\frac{M^2}{\mu^2}\right) + \ln\left(\frac{(1-\hat{\tau})^2}{\hat{\tau}}\right) \right] + \frac{1}{4} + \frac{3}{2}\hat{\tau} - \frac{7}{4}\hat{\tau}^2 \right\} \end{aligned}$$

- Similarly,

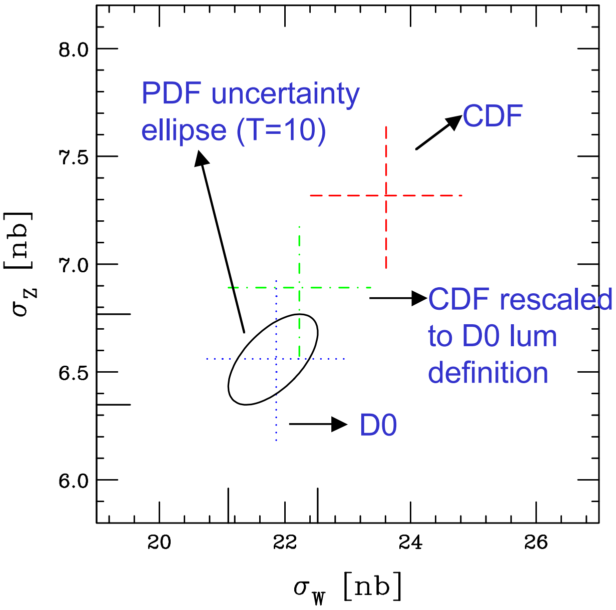
$$\begin{aligned} H_{G\bar{q}'}^{(1)} &= \sigma_{G\bar{q}'}^{(1)} - \left[\phi_{q\leftarrow G}^{(1)} \sigma_{q\bar{q}'}^{(0)}\right] \\ &= H_{qG}^{(1)} \end{aligned}$$

(Note: If we choose the renormalization scale $\mu^2 = M^2$, then $\ln\left(\frac{M^2}{\mu^2}\right) = 0$)

W and Z production

- * CDF and D0 would like to use their W and Z cross sections for luminosity determination
- * D0 cross sections close to center of PDF prediction ellipse; not the case with CDF

MSU study



J. Pumplin, D. Stump, R. Brock, D. Casey, J. Huston, J. Kalk, H.L. Lai, W.K. Tung: hep-ph/0101051

Summary

- $\phi_{f/h}(x,\mu^2)$ depends on scheme ($\overline{MS}$,DIS,...)
  $\Rightarrow H_{ij}$ **scheme dependent**

- Evolution equations allow us to perdict
  q^2-**dependent of** $\phi(x,q^2)$

- Essentially identical procedure for
  $hh' \to jets$, inclusive $Q\overline{Q}$,...
  But, when the Born level process involves
  strong interaction (eg. $q\bar{q} \to t\bar{t}$),
  it is also necessary to renormalize the
  strong coupling α_s, etc, to eliminate
  ultraviolate singularities

Appendix A
γ-matrices in n dimensions

- Dirac algebra

$$
\begin{aligned}
\{\gamma^\mu, \gamma^\nu\}_+ &\equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \\
\mu, \nu &= 1, 2, \cdots, n \qquad g^{\mu\nu} = diag\,(1, -1, \cdots, -1) \\
g^{\mu\nu}g_{\mu\nu} &= n \\
\{\gamma^\mu, \gamma^5\}_+ &= 0 \qquad (\text{Naive-}\gamma^5\text{prescription})
\end{aligned}
$$

This works in calculating the inclusive rate of W-boson , but fails in the differential distributions of the leptons from the W-boson decay.

- Matrix identities

$n = 4 - 2\varepsilon$

$$
\begin{aligned}
\gamma_\mu \not{a}\gamma^\mu &= -2\,(1-\varepsilon)\not{a} \\
\gamma_\mu \not{a}\not{b}\gamma^\mu &= 4a\cdot b - 2\varepsilon\not{a}\not{b} \\
\gamma_\mu \not{a}\not{b}\not{c}\gamma^\mu &= -2\not{c}\not{b}\not{a} + 2\varepsilon\not{a}\not{b}\not{c}
\end{aligned}
$$

- Traces

$$
\begin{aligned}
Tr\,[\not{a}\not{b}] &= 4\,(a\cdot b) \\
Tr\,[\not{a}\not{b}\not{c}\not{d}] &= 4\,\{(a\cdot b)\,(c\cdot d) - (a\cdot c)\,(b\cdot d) + (a\cdot d)\,(b\cdot c)\} \\
Tr\,[\gamma_5\not{a}\not{b}] &= 0
\end{aligned}
$$

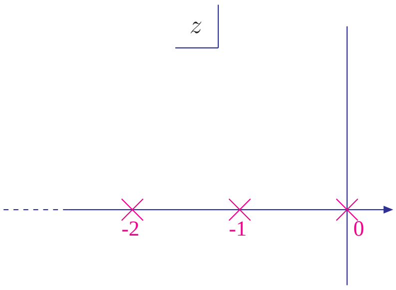
Appendix B
Some integrals and ”special functions”

- The ”Gamma function”

$$\Gamma\left(z\right) = \int_0^{\infty} dx x^{z-1} e^{-x} \qquad (\mathrm{Re}\left(z\right) > 0)$$

$$\Gamma\left(z-1\right) = \frac{\Gamma\left(z\right)}{z-1} \qquad \text{(for all z)}$$

$\Rightarrow$ Poles in $\Gamma(z)$



$$\Gamma\left(n\right) = \left(n-1\right)! \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(\varepsilon\right) = \frac{1}{\varepsilon} - \gamma_E + \frac{\varepsilon}{2}\left(\gamma_E^2 + \frac{\pi^2}{6}\right) + \cdots$$

$$(\gamma_E = 0.5772\cdots, \text{ Euler constant})$$

$$\Gamma\left(1-\varepsilon\right) = -\varepsilon\Gamma\left(\varepsilon\right) = 1 + \varepsilon\gamma_E + \frac{1}{2}\varepsilon^2\left(\frac{\pi^2}{6} + \gamma_E^2\right) + O\left(\varepsilon^3\right)$$

$$\Gamma\left(1-\varepsilon\right)\Gamma\left(1+\varepsilon\right) = 1 + \varepsilon^2\frac{\pi^2}{6} + O\left(\varepsilon^4\right)$$

$$z^{\varepsilon} = e^{\ln z^{\varepsilon}} = e^{\varepsilon \ln z} = 1 + \varepsilon \ln z + \cdots$$

- The ”Beta function”

$$B\left(\alpha,\beta\right) = \frac{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}{\Gamma\left(\alpha+\beta\right)}$$

$$B\left(\alpha,\beta\right) = \int_0^1 dy\; y^{\alpha-1}\left(1-y\right)^{\beta-1} = \int_0^{\infty} dy\; y^{\alpha-1}\left(1+y\right)^{-\alpha-\beta}$$

$$= 2\int_0^{\frac{\pi}{2}} d\theta\; \left(\sin\theta\right)^{2\alpha-1}\left(\cos\theta\right)^{2\beta-1}$$

- Feynman trick

$$\frac{1}{ab} = \int_0^1 dx \frac{1}{[ax + b(1-x)]^2}$$

$$\frac{1}{a^\alpha b^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \frac{x^{\alpha-1}(1-x)^{\beta-1}}{[ax + b(1-x)]^{\alpha+\beta}}$$

- n-dimension integrals

$$\int d^n l \frac{l_\mu}{(l^2 - M^2)^\alpha} = 0$$

$$\int d^n l \frac{l_\mu l_\nu}{(l^2 - M^2)^\alpha} = \int d^n l \frac{\left(\frac{l^2 g_{\mu\nu}}{n}\right)}{(l^2 - M^2)^\alpha}$$

$$\int \frac{d^n l}{(2\pi)^n} \frac{1}{(l^2 - M^2)^\alpha} = i \frac{(-1)^\alpha}{(4\pi)^{n/2}} \frac{\Gamma\left(\alpha - \frac{n}{2}\right)}{\Gamma(\alpha)} \left(\frac{1}{M^2}\right)^{\alpha - \frac{n}{2}}$$

$$\int d^n l \frac{l^2}{(l^2 - M^2)^\alpha} = \int d^n l \frac{(l^2 - M^2) + M^2}{(l^2 - M^2)^\alpha}$$

- $$\mathrm{Re}\left[(-1)^\varepsilon\right] = 1 - \varepsilon^2 \frac{\pi^2}{2} + O\left(\varepsilon^4\right)$$

- "plus distribution"— to isolate $\frac{1}{\varepsilon}$ poles

Consider $\dfrac{1}{(1-z)^{1+2\varepsilon}}$

$$= \frac{1}{(1-z)^{1+2\varepsilon}} - \left[\delta(1-z)\int_0^1 \frac{dz'}{(1-z')^{1+2\varepsilon}} + \frac{1}{2\varepsilon}\delta(1-z)\right]$$

$$\searrow \quad \swarrow$$

cancel

because $\displaystyle\int_0^1 \frac{dz'}{(1-z')^{1+2\varepsilon}} = \frac{-1}{2\varepsilon}$ for $\varepsilon \to 0^-$

$$\equiv \left[\frac{1}{(1-z)^{1+2\varepsilon}}\right]_+ - \frac{1}{2\varepsilon}\delta(1-z)$$

$$= \frac{1}{(1-z)_+} - 2\varepsilon\left[\frac{\ln(1-z)}{1-z}\right]_+ + O\left(\varepsilon^2\right) - \frac{1}{2\varepsilon}\delta(1-z)$$

because $\displaystyle\frac{1}{(1-z)^{2\varepsilon}} = (1-z)^{-2\varepsilon}$

$$= 1 - 2\varepsilon\ln(1-z) + \cdots$$

- $[\cdots]_+$ is a distribution

$$\int_0^1 dz\ f(z)\left[\frac{1}{1-z}\right]_+$$

$$\equiv \int_0^1 dz\,\frac{f(z)}{1-z} - \int_0^1 dz\ f(z)\,\delta(1-z)\int_0^1 \frac{dz'}{(1-z')}$$

$$= \int_0^1 dz\frac{f(z)-f(1)}{1-z},$$ which is finite.

Appendix C
Angular integrals in n dimensions

- In n dimensions

$$\int d^n x = \int r^{n-1} d\Omega_{n-1}$$

-

$$\int d\Omega_n = \int_0^\pi d\theta_{n-1} \sin^{n-1}\theta_{n-1} \int_0^\pi d\theta_{n-2} \sin^{n-2}\theta_{n-2} \cdots \int_0^\pi d\theta_1 \sin\theta_1 \int_0^{2\pi} d\phi$$

$$\Rightarrow \int d\Omega_1 = \int_0^{2\pi} d\phi \qquad \longrightarrow \Omega_1 = 2\pi$$

$$\int d\Omega_2 = \int_0^\pi d\theta_1 \sin\theta_1 \int d\Omega_1 \qquad \longrightarrow \Omega_2 = 4\pi$$

$$\vdots$$

$$\int d\Omega_n = \int_0^\pi d\theta_{n-1} \sin^{n-1}\theta_{n-1} \int d\Omega_{n-1}$$

$$\Rightarrow \qquad \Omega_n = \frac{2^n \pi^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)}{\Gamma(n)}$$ from repeated use of $B(\alpha, \beta)$

$$= \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}$$ because $\Gamma(n) = \frac{2^{n-1}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$

Appendix D

Two-particle phase space in n dimensions

- $$\int_{PS_2(p)} dk\,dq = \int \frac{d^{n-1}\vec{k}}{(2\pi)^{n-1}2k_0}\frac{d^{n-1}\vec{q}}{(2\pi)^{n-1}2q_0}\cdot(2\pi)^n\,\delta^n\left(p-q-k\right)$$

  with $k^\mu = \left(k_0,\vec{k}\right)$, etc.

  Use $\frac{d^{n-1}\vec{q}}{2q_0} = \int d^nq\delta^+\left(q^2-Q^2\right)$, we get

- $$\int_{PS_2(p)} dk\,dq = \frac{1}{(2\pi)^{n-2}}\int\frac{d^{n-1}\vec{k}}{2k_0}\delta^+\left((p-k)^2-Q^2\right)$$
  $$= \frac{1}{(2\pi)^{n-2}}\int\frac{dk\,k^{n-3}}{2}\int d\Omega_{n-2}\delta\left(\hat{s}-2k\sqrt{\hat{s}}-Q^2\right)$$
  $$\left(p^2\equiv\hat{s},k^2=0,k=\left|\vec{k}\right|\right)$$

  Use $n=4-2\varepsilon$, then in the c.m. frame $\left(p^\mu=\left(\sqrt{\hat{s}},\vec{0}\right)\right)$,

- $$\int_{PS_2(p)} dk\,dq = \frac{\Omega_{n-3}}{(2\pi)^{2(1-\varepsilon)}}\int\frac{dk\,k^{1-2\varepsilon}}{4\sqrt{\hat{s}}}\int_0^\pi d\theta\,(\sin\theta)^{1-2\varepsilon}\cdot\delta\left(k-\frac{\hat{s}-Q^2}{2\sqrt{\hat{s}}}\right)$$

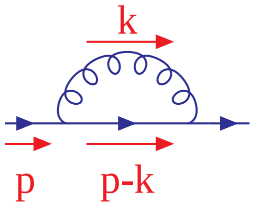
  Use new variables:

  $$z=\frac{Q^2}{\hat{s}},y=\frac{1}{2}(1+\cos\theta)\Rightarrow k=\frac{\sqrt{\hat{s}}}{2}(1-z),$$

- $$\int_{PS_2(p)} dk\,dq = \frac{1}{8\pi}\left(\frac{4\pi}{Q^2}\right)^\varepsilon\frac{z^\varepsilon(1-z)^{1-2\varepsilon}}{\Gamma(1-\varepsilon)}\int_0^1 dy\,[y(1-y)]^{-\varepsilon}$$

Appendix E
Explicit Calculations

- Consider



$$\int \frac{d^n k}{(2\pi)^n} \frac{\gamma_\mu \left(\not{p} - \not{k} \right) \gamma^\mu}{\left(k^2 + i\epsilon\right) \left((p-k)^2 + i\epsilon\right)}$$

$$\rightarrow \int \frac{d^n k}{(2\pi)^n} \int_0^1 dx \frac{(2-n)\left(\not{p} - \not{k} \right)}{\left[k^2 - 2k \cdot xp\right]^2} \qquad (l \equiv k - xp)$$

$$= \int \frac{d^n l}{(2\pi)^n} \int_0^1 dx \frac{(2-n)\left[(1-x)\ \not{p} - \not{l}\right]}{\left[l^2 + i\epsilon\right]^2}$$

$$= \left[\left(1 - \frac{n}{2}\right)\ \not{p} \right] \cdot \underbrace{\int \frac{d^n l}{(2\pi)^n} \frac{1}{\left[l^2 + i\epsilon\right]^2}}$$

$$\downarrow$$

$$= 0 \quad \left(\begin{array}{c} \text{Because there is} \\ \text{no mass scale} \end{array} \right)$$

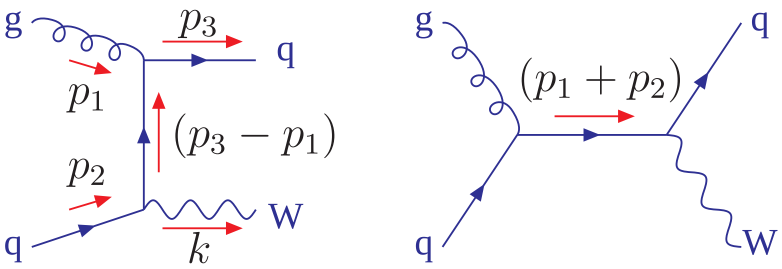
$$\uparrow$$

$$\left(\begin{array}{c} \text{Due to cancellation} \\ \text{of } \frac{1}{\varepsilon_{UV}} \text{ and } \frac{1}{\varepsilon_{IR}} \\ \text{Trick:} A = A - B + B \end{array} \right)$$

$$= \int \frac{d^n l}{(2\pi)^n} \left\{ \underbrace{\left[\frac{1}{\left(l^2\right)^2} - \frac{1}{\left(l^2 - \Lambda^2\right)^2} \right]}_{\text{IR div.}} + \underbrace{\left[\frac{1}{\left(l^2 - \Lambda^2\right)^2} \right]}_{\text{UV div.}} \right\}$$

$$= \frac{i}{16\pi^2} \left(\frac{1}{\varepsilon_{IR}} \right) + \frac{i}{16\pi} \left(\frac{1}{\varepsilon_{UV}} \right), \qquad \left(\begin{array}{c} n - 4 = 2\varepsilon_{IR} \\ 4 - n = 2\varepsilon_{UV} \end{array} \right)$$

- consider the real emission process



Define the Mandelstam variables

$$\begin{aligned}\hat{s} &= (p_1+p_2)^2 = \ \ 2p_1\cdot p_2\\ \hat{t} &= (p_1-p_3)^2 = -2p_1\cdot p_3\\ \hat{u} &= (p_2-p_3)^2 = -2p_2\cdot p_3\end{aligned}$$

After averaging over colors and spins

$$\overline{|\mathcal{M}|^2} = \underbrace{\left(\frac{1}{2\,(1-\varepsilon)}\frac{1}{2}\right)}_{\text{Spin}} \cdot \underbrace{\left(\frac{1}{3}\cdot\frac{1}{8}\right)\cdot Tr\,(t^a t^a)}_{\text{Color}} \cdot (g\mu^{\varepsilon})^2$$

$$\cdot\, g_w^2 \cdot 2\,(1-\varepsilon)$$

$$\cdot \left[(1-\varepsilon)\left(-\frac{\hat{s}}{\hat{t}}-\frac{\hat{t}}{\hat{s}}\right)-\frac{2\hat{u}M^2}{\hat{t}\hat{s}}+2\varepsilon\right]$$

Note: The d.o.f. of gluon polarization is $2\,(1-\varepsilon)$, and that of quark polarization is 2.

- In the parton c.m. frame, the constituent cross section

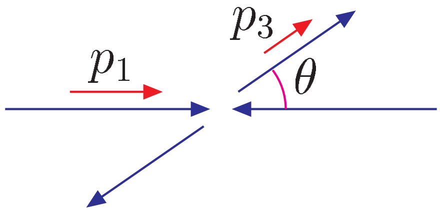
$$\hat{\sigma} = \frac{1}{2\hat{s}}\overline{|\mathcal{M}|^2} \cdot (PS_2)$$

$$= \frac{1}{2\hat{s}} \cdot \left\{ \frac{1}{4} \cdot \frac{1}{6} \cdot 2 g_s^2 \mu^{2\varepsilon} g_w^2 (1-\varepsilon) \cdot \right.$$

$$\left. \left[(1-\varepsilon)\left(-\frac{\hat{s}}{\hat{t}} - \frac{\hat{t}}{\hat{s}} \right) - \frac{2\hat{u}M^2}{\hat{t}\hat{s}} + 2\varepsilon \right] \right\}$$

$$\cdot \left\{ \frac{1}{8\pi} \left(\frac{4\pi}{M^2} \right)^{\varepsilon} \frac{1}{\Gamma(1-\varepsilon)} \hat{\tau}^{\varepsilon} (1-\hat{\tau})^{1-2\varepsilon} \int_0^1 dy \, [y(1-y)]^{-\varepsilon} \right\}$$

where $y \equiv \frac{1}{2}(1+\cos\theta)$



Using $\hat{t} = -\hat{s}\left(1 - \frac{M^2}{\hat{s}}\right)(1-y)$

$$\hat{u} = -\hat{s}\left(1 - \frac{M^2}{\hat{s}}\right) y$$

and

$$\int_0^1 dy \, y^{\alpha} (1-y)^{\beta} = \frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{\Gamma(2+\alpha+\beta)} ,$$

we get

$$\hat{\sigma}_{qG} = \hat{\sigma}^{(0)} \frac{\alpha_s}{4\pi} \cdot \left\{ 2P^{(1)}_{q\leftarrow g}(\hat{\tau}) \left[\frac{-1}{\varepsilon} \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} + \ln \frac{M^2(1-\hat{\tau})^2}{4\pi\mu^2\hat{\tau}} \right] \right.$$

$$\left. + \frac{3}{2} + \hat{\tau} - \frac{3}{2}\hat{\tau}^2 \right\},$$

with

$$P^{(1)}_{q\leftarrow g}(\hat{\tau}) = \frac{1}{2}\left[\hat{\tau}^2 + (1-\hat{\tau})^2 \right]$$

$$\hat{\sigma}^{(0)} \equiv \frac{\pi}{12} g_w^2 \frac{1}{\hat{s}}$$