Higher Order Tools, part I

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2011 CTEQ Summer School, UW-Madison

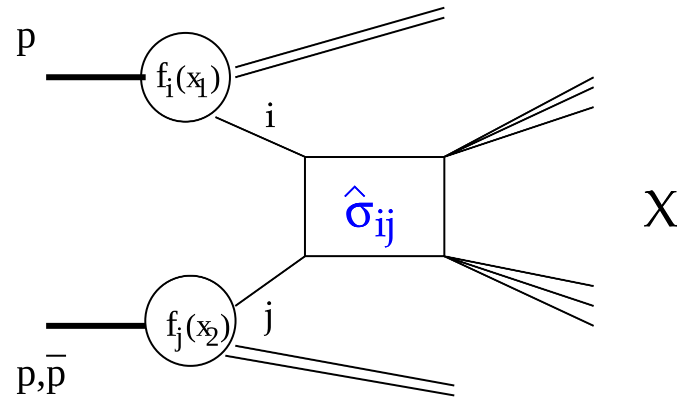
Outline of this lecture

- Setting the frame: basic concepts and terminology.
- Structure of a Next-to-Leading (NLO) calculation:
  $\longrightarrow$ virtual corrections: Feynman diagram approach versus new techniques based on generalized unitarity and on-shell recursion relations;
  $\longrightarrow$ real corrections: isolating IR divergences and matching with virtual corrections and parton densities.
- Examples of NLO results: what do we gain?
- When is NLO not enough? Examples of physical observables that require various levels of improvement:
  $\longrightarrow$ one more order in the perturbative expansion: going Next-to-Next-to-Leading Order (NNLO);
  $\longrightarrow$ resummation of large corrections at all orders: reordering the perturbative expansion.

Setting the Frame

- Hadron colliders (Tevatron, LHC) are the present and close future of particle physics: emphasis on QCD ($\rightarrow$ see G. Sterman's lectures).
- We will learn about the properties of NLO calculations by considering:
  - $\longrightarrow$ prototype process: QCD top quark pair production, $q\bar{q}, gg \rightarrow t\bar{t}$ ($\rightarrow$ see F. Olness's lectures on Heavy Quarks).
  - $\longrightarrow$ first order of QCD corrections;
  - $\longrightarrow$ total/differential cross-sections.
- I will assume a basic knowledge of some fundamental topics of Quantum Field Theory as encountered in Quantum Electrodynamics:
  - $\longrightarrow$ perturbative calculation of cross-section from Feynman diagrams;
  - $\longrightarrow$ origin of ultraviolet (UV) and infrared (IR) divergences;
  - $\longrightarrow$ regularization and renormalization of UV divergences;
  - $\longrightarrow$ cancellation of IR divergences for IR-safe observables.

The basic picture of a $p\bar{p}, pp \to X$ high energy process is ...



where the short and long distance part of the QCD interactions can be factorized and the cross section for $pp, p\bar{p} \to X$ can be calculated as:

$$d\sigma(pp, p\bar{p} \to X) = \sum_{ij} \int dx_1 dx_2 f_i{}^p(x_1, \mu_F) f_j{}^{p,\bar{p}}(x_2, \mu_F) d\hat{\sigma}(ij \to X, x_1, x_2, Q^2, \mu_F)$$

$\longrightarrow$ $ij \to$ quarks or gluons (partons)

$\longrightarrow$ $f_i{}^p(x)$, $f_i{}^{p,\bar{p}}(x)$: Parton Distributions Functions (PDF)

($x \to$ fraction of hadron momentum carried by parton i)

$\longrightarrow$ $d\hat{\sigma}(ij \to X)$: partonic cross section

$\longrightarrow$ μ_F: factorization scale.

$\longrightarrow$ Q^2: hard scattering scale.

$\longrightarrow$ In the $ij \to X$ process, initial and final state partons radiate and absorb gluons/quarks (both real and virtual).

$\longrightarrow$ Due to the very same interactions: the strong coupling constant $(\alpha_s = g_s^2/4\pi)$ becomes small at large energies (Q^2):

$$\alpha_s(Q^2) \to 0 \quad \text{for large scales } Q^2 \quad : \quad \text{asymptotic freedom}$$

$\longrightarrow$ We can calculate $d\hat{\sigma}(ij \to X)$ perturbatively:

$$d\hat{\sigma}(ij \to X) = \alpha_s^k \sum_{m=0}^{\infty} d\hat{\sigma}_{ij}^{(m)} \alpha_s^m$$

n=0 : Leading Order (LO), or tree level or Born level

n=1 : Next to Leading Order (NLO), includes $O(\alpha_s)$ corrections

$\ldots\ldots$

$\longrightarrow$ We can calculate the evolution of the PDF's perturbatively (from DGLAP equation): LO, NLO, ... PDF's.

Perturbative approach and scale dependence ...

$\longrightarrow$ At each order in α_s both partonic cross section and PDF's have a residual factorization scale dependence (μ_F).

$\longrightarrow$ At each order in α_s the expression of $\hat{\sigma}(ij \to X)$ contains UV infinities that are renormalized. A remnant of the subtraction point is left at each perturbative order as a renormalization scale dependence (μ_R)

$$d\hat{\sigma}(ij \to X, Q^2, \mu_F) = \alpha_s^k(\mu_R) \sum_{m=0}^{\infty} d\hat{\sigma}_{ij}^{(m)}(Q^2, \mu_F, \mu_R)\alpha_s^m(\mu_R)$$

Setting $\boxed{\mu_R = \mu_F = \mu}$ (often adopted simplifying assumption):

$$d\sigma = \sum_{ij} \int dx_1 dx_2 f_i^p(x_1, \mu) f_j^{p,\bar{p}}(x_2, \mu) \sum_{m=0}^{\infty} d\hat{\sigma}_{ij}^{(m)}(x_1, x_2, Q^2, \mu)\alpha_s^{m+k}(\mu)$$

The residual scale dependence should improve with the perturbative order

General structure of a NLO calculation

NLO cross-section:

$$d\sigma^{NLO}_{p\bar{p},pp} = \sum_{i,j} \int dx_1 dx_2 f_i^p(x_1,\mu) f_j^{\bar{p},p}(x_2,\mu) d\hat{\sigma}^{NLO}_{ij}(x_1,x_2,\mu)$$

where

$$d\hat{\sigma}^{NLO}_{ij} = d\hat{\sigma}^{LO}_{ij} + \frac{\alpha_s}{4\pi}\delta d\hat{\sigma}^{NLO}_{ij}$$

NLO corrections made of:

$$\delta d\hat{\sigma}^{NLO}_{ij} = d\hat{\sigma}^{virt}_{ij} + d\hat{\sigma}^{real}_{ij}$$

- $d\hat{\sigma}^{virt}_{ij}$: one loop virtual corrections.
- $d\hat{\sigma}^{real}_{ij}$: one gluon/quark real emission.
- use $\alpha_s^{NLO}(\mu)$ and match with NLO PDF's.

$\longrightarrow$ renormalize UV divergences ($d=4-2\epsilon_{UV}$)

$\longrightarrow$ cancel IR divergences in $d\hat{\sigma}^{virt}_{ij} + d\hat{\sigma}^{real}_{ij}$ + PDF's ($d=4-2\epsilon_{IR}$)

$\longrightarrow$ check μ-dependence of $d\sigma^{NLO}_{p\bar{p},pp}$ (μ_R,μ_F)

On the issue of scale dependence ...

At a given order, the dependence on the renormalization/factorization scales is a higher order effect.

Let us rewrite $d\hat{\sigma}_{ij}^{NLO}(x_1,x_2,\mu)$ making the scale dependence explicit:

$$d\hat{\sigma}_{ij}^{NLO}(x_1,x_2,\mu) = \alpha_s^k(\mu)\left\{\mathcal{F}_{ij}^{LO}(x_1,x_2) + \frac{\alpha_s(\mu)}{4\pi}\left[\mathcal{F}_{ij}^{1}(x_1,x_2) + \bar{\mathcal{F}}_{ij}^{1}(x_1,x_2)\ln\left(\frac{\mu^2}{\hat{s}}\right)\right]\right\}$$

$\bar{\mathcal{F}}_{ij}^{1}(x_1,x_2)$ can be calculated by imposing that the hadronic cross-section is scale independent at the perturbative order of the calculation (NLO), i.e.:

$$\mu^2\frac{d}{d\mu^2}d\sigma_{p\bar{p},pp}^{NLO} = \mathcal{O}(\alpha_s^{(k+2)})$$

Using that $\mu^2\frac{d}{d\mu^2}\alpha_s(\mu) = -b_0\alpha_s^2+\cdots$, and the DGLAP equation for the scale evolution of PDF's, one gets:

$$\begin{aligned}\bar{\mathcal{F}}_{ij}^{1}(x_1,x_2) &= 2\,\Big\{4\pi b_0\mathcal{F}_{ij}^{LO}(x_1,x_2)\\ &- \sum_k\left[\int_\rho^1 dz_1 P_{ik}(z_1)\mathcal{F}_{kj}^{LO}(x_1z_1,x_2) + \int_\rho^1 dz_2 P_{jk}(z_2)\mathcal{F}_{ik}^{LO}(x_1,x_2z_2)\right]\Big\}\end{aligned}$$

($P_{ij} \longrightarrow$ Altarelli-Parisi splitting functions).

Approaching virtual corrections ...

The $\mathcal{O}(\alpha_s)$ virtual corrections to the cross-section arise from the interference between the tree level amplitude $\mathcal{A}_0(ij \to \{f\})$ and the one-loop virtual amplitude $\mathcal{A}_1^{virt}(ij \to \{f\})$:

$$d\hat{\sigma}_{ij}^{virt} = \int d(PS_{\{f\}}) \overline{\sum} 2Re(\mathcal{A}_1^{virt}\mathcal{A}_0^*)$$

$\mathcal{A}_0$ and $\mathcal{A}_1^{virt}$ can be further organized in terms of

- color structures: color ordered amplitudes (primitive amplitudes);
- helicity/spin amplitudes (massless/massive particles);

and calculated using different methods:

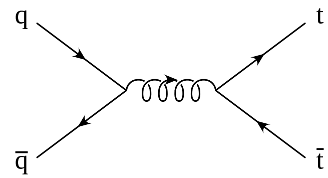
$\longrightarrow$ Feynman diagrams: direct perturbative expansion of S-matrix elements (off-shell intermediate states in loops);

$\longrightarrow$ Unitarity methods: determine scattering amplitudes from their analytical poles and cuts (on-shell intermediate states in loops).

Example: $p\bar{p}, pp \to t\bar{t}$, tree level

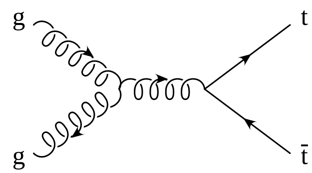
$q\bar{q} \to t\bar{t}$

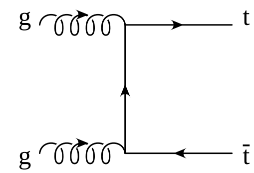
leading contribution at the Tevatron

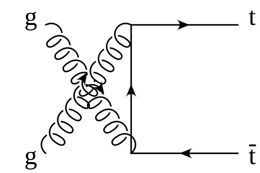


$gg \to t\bar{t}$

leading contribution at the LHC







NLO corrections calculated in:

→ P. Nason, S. Dawson, R.K. Ellis, NPB 303 (1988) 607, NPB 327 (1989) 49;

→ W. Beenakker, H. Kuijf, W.L. van Neerven, J. Smith PRD 40 (1989) 54; (with R. Meng and G.A. Schuler) NPB 351 (1991) 507.

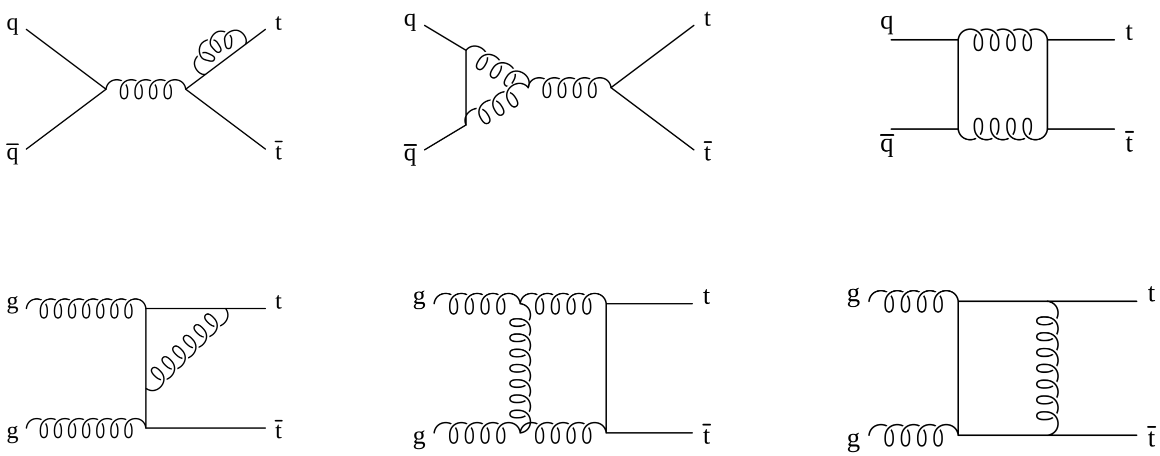
NNLO corrections in progress:

→ M. Czakon, A. Mitov, S. Moch, PLB 651 (2007) 174, NPB 798 (2008) 210 M. Czakon, A. Mitov, arXiv:0811.4119.

→ R. Bonciani, et al., arXiv:0806.2301, arXiv:0906.3671, arXiv:1011.6661

$p\bar{p}, pp \to t\bar{t}$: $\mathcal{O}(\alpha_s)$ virtual corrections using Feynman diagrams

$\mathcal{A}_1^{virt}(q\bar{q}, gg \to t\bar{t})$ receives contributions from self-energy, vertex, and box type loop corrections:



most of which contains UV and IR divergences that need to be extracted analytically (in $D = 4 - 2\epsilon$).

One can isolate color ordered amplitudes observing that:

$$\mathcal{A}_0(q\bar{q} \to t\bar{t}) = \mathcal{A}_0^{q\bar{q}} t^a \times t^a$$

$$\mathcal{A}_1(q\bar{q} \to t\bar{t}) = \mathcal{A}_1^{q\bar{q}(1)} t^a t^b \times t^a t^b + \mathcal{A}_1^{q\bar{q}(2)} t^a t^b \times t^b t^a$$

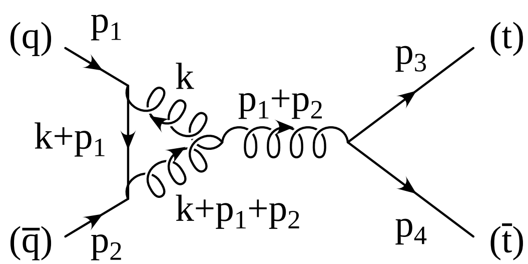
$$\mathcal{A}_{0,1}(gg \to t\bar{t}) = \mathcal{A}_{0,1}^{gg(1)} t^a t^b + \mathcal{A}_{0,1}^{gg(2)} t^b t^a$$

and introduce helicity/spin states if desired.

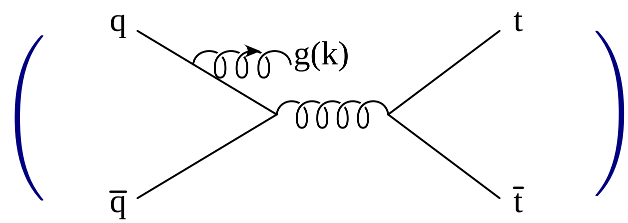
For each diagram:

- write the corresponding amplitude and reduce the Dirac's algebra to fundamental structures;
- the coefficient of each Dirac's structure contains both scalar and tensor integrals of the loop momentum;
- tensor integrals can be reduced to scalar integrals;
- scalar integrals are computed and UV and IR divergences are extracted using dimensional regularization in $d = 4 - 2\epsilon$;
- UV divergences (two- and three-point functions) are extracted as poles in $\frac{1}{\epsilon_{UV}}$, and subtracted using a given renormalization scheme ($\overline{MS}$, on-shell, etc.);
- IR divergences (two-, three- and four-point functions) are extracted as poles in $\frac{1}{\epsilon^2_{IR}}$ and $\frac{1}{\epsilon_{IR}}$ $\longrightarrow$ cancelled in $d\hat{\sigma}_{ij}^{virt} + d\hat{\sigma}_{ij}^{real}$ + PDF's.

Example. Consider the vertex correction:



$\leftrightarrow$



the (unpolarized) amplitude associated to this diagram is of the form:

$$\mathcal{A} \propto \int \frac{d^d k}{(2\pi)^d} \frac{\bar{v}(p_2)\gamma_\rho(\not{k}+\not{p}_1)\gamma_\nu u(p_1)}{k^2(k+p_1)^2(k+p_1+p_2)^2} \frac{\bar{u}(p_3)\gamma_\mu v(p_4)}{(p_1+p_2)^2} V^{\mu\nu\rho}(-p_1-p_2,-k,k+p_1+p_2)$$

and depends on the following scalar/tensor integrals:

$$C_0, C_1^\mu, C_2^{\mu\nu}(p_1,p_2) = \int \frac{d^d k}{(2\pi)^d} \frac{1, k^\mu, k^\mu k^\nu}{k^2(k+p_1)^2(k+p_1+p_2)^2} = \int \frac{d^d k}{(2\pi)^d} \frac{1, k^\mu, k^\mu k^\nu}{D_3(p_1,p_2)}$$

$C_2^{\mu\nu}$ is UV divergent, while all of them are IR divergent, as can be easily recognized by simple power counting and observing that

$$D_3(p_1,p_2) \xrightarrow{k\to k-p_1} k^2(k-p_1)^2(k+p_2)^2 \xrightarrow{k^2\simeq 0} k^2(k\cdot p_1)(k\cdot p_2)$$

- $k^0 \to 0$: soft divergence;
- $k\cdot p_1 \to 0$ or $k\cdot p_2 \to 0$: collinear divergence.

We can parametrize C_1^μ and $C_2^{\mu\nu}$ as:

$$
\begin{aligned}
C_1^\mu(p_1,p_2) = \int \frac{d^dk}{(2\pi)^d}\frac{k^\mu}{D_3(p_1,p_2)} &= C_1^1 p_1^\mu + C_1^2 p_2^\mu \\
C_2^{\mu\nu}(p_1,p_2) = \int \frac{d^dk}{(2\pi)^d}\frac{k^\mu k^\nu}{D_3(p_1,p_2)} &= C_2^{00} g^{\mu\nu} + C_2^{11} p_1^\mu p_1^\nu + C_2^{22} p_2^\mu p_2^\nu + \\
& \quad C_2^{12}(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu)
\end{aligned}
$$

The tensor integral coefficients can be obtained using different methods. In this case the easiest way is by saturating the independent tensor structures: Passarino-Veltman method (NPB 160 (1979) 151).

<u>Ex</u>: C_1^1 and C_1^2 are very simple:

$$
\begin{aligned}
C_1^1 &= \frac{1}{2p_1\cdot p_2}\left(B0_{12} - B0_{13} - 2p_1\cdot p_2 C_0\right) \\
C_1^2 &= \frac{1}{2p_1\cdot p_2}\left(B0_{13} - B0_{23}\right)
\end{aligned}
$$

where $B0_{ij}$ are scalar integrals with two (out of the three) denominators of C_0.

Ultimately, everything is expressed in terms of B_0, and C_0 scalar integrals.

Introducing the appropriate Feynman's parameterization the diagram in question can be written as:

$$\Gamma(3) \int_0^1 dx\, x \int_0^1 dy \int \frac{d^d k'}{(2\pi)^d} \frac{\mathrm{Num}(k \to k')}{[(k')^2 - \Delta]^3}$$

where $(k')^\mu = (k - x(1-y)p_1 + (1-x)p_2)^\mu$ and $\Delta = -x(1-x)(1-y)2p_1 \cdot p_2$.

Using standard d-dimensional integrals:

$$\int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 - \Delta)^n} = i \frac{(-1)^n}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)} \frac{1}{\Delta^{n-d/2}}$$

and upon integration over the Feynman parameters one gets (including couplings and color factor):

$$\frac{\alpha_s}{4\pi} \left(\frac{4\pi\mu}{\hat{s}}\right)^\epsilon \frac{N}{2} \Gamma(1+\epsilon) \left(-\frac{4}{\epsilon_{IR}} + \frac{3}{\epsilon_{UV}} - 2\right) \mathcal{A}_0(q\bar{q} \to t\bar{t})$$

This is a very simple case, but more complex ones are solved using the same strategy.

Passarino-Veltman reduction can be problematic when considering high rank tensor integrals in processes with more external particles.
The tensor integrals coefficients are proportional to inverse powers of the Gram determinant (GD):

$$GD = \det(p_i \cdot p_j) \qquad (p_i, p_j \to \text{independent external momenta})$$

(where $GD = (p_1 \cdot p_2)^2$ in the example we saw.) For instance, for a $2 \to 3$ process:

$$GD(p_1 + p_2 \to p_3 + p_4 + p_5) \simeq f(E_3, E_4, \sin\theta_3, \sin\theta_4, \sin\phi_4)$$

$\boxed{GD \to 0}$ when two momenta become degenerate: spurious divergences that creates numerical instabilities.

Possible alternatives:

$\longrightarrow$ Eliminate all dangerous tensor integrals at the level of the amplitude square, if possible.

$\longrightarrow$ Kinematic cuts to avoid numerical instabilities and extrapolation to the unsafe region using several algorithms.

$\longrightarrow$ Expansion of coefficients about limit of vanishing kinematic determinants, including $GD \to 0$ ($\to$ see, e.g., S. Dittmaier, A. Denner, NPB 734 (2006) 62)

$p\bar{p}, pp \to t\bar{t}$: $\mathcal{O}(\alpha_s)$ virtual corrections using generalized unitarity and analytical properties of 1-loop amplitude

As we have seen any color ordered $\mathcal{A}_1(q\bar{q}, gg \to t\bar{t})$ reduces to a linear combination of scalar integrals of the form:

$$\mathcal{A}_1 = \sum_i d_i D0_i + \sum_i c_i C0_i + \sum_i b_i B0_i + \sum_i a_i A0_i + R_1$$

This is very general, and applies to any one-loop QCD amplitude.

- The (d-dimensional) scalar integrals $D0_i$, $C0_i$, $B0_i$, $A0_i$ are known (collected in: R. K. Ellis and G. Zanderighi, JHEP 0802:002 2008).
- The coefficients of the scalar integrals can be evaluated using generalized unitarity methods.
- The rational terms R_1 can be computed based on the analytic properties of the amplitude or using d-dimensional generalized unitarity.

Z. Bern, L. Dixon, D. Forde, D. Kosower, et al.
R. K. Ellis, W. Giele, Z. Kunzst, K. Melnikov, G. Zanderighi
G. Ossola, C. G. Papadopoulos, R. Pittau
R. Britto, F. Cachazo, B. Feng

Coefficients $d_i, \ldots, a_i$ from generalized unitarity

From the unitarity of the scattering matrix ($S = 1 + iT$) one derives that:

$$-i(T - T^\dagger) = T^\dagger T$$

or equivalently:

$$2\,\mathrm{Im}\,\mathcal{A}(i \to f) = \sum_n \int d\Pi_n \mathcal{A}^*(i \to n)\mathcal{A}(f \to n)$$

The l.h.s. represents the discontinuity of $\mathcal{A}(s_{ij})$ across the branch cut corresponding to the invariant s_{ij}, for each s_{ij}.

Diagramatically this can be calculated by replacing the two propagators connecting a set of legs carrying that invariant to the rest of the diagram by on-shell delta functions ($\to$ Cutkosky rule), e.g.

$$\frac{i}{p^2 + i\epsilon} \to \pi\delta^+(p^2) \quad \longrightarrow \quad \text{cut}$$

Example: for a discontinuity in the s_{12} channel of a $2 \to 2$ process ($k_1 = k - k_2$)

$$2\,\mathrm{Im}\,\mathcal{A}_1(1{+}2 \to 3{+}4)|_{12} = \int \frac{d^4k}{(2\pi)^4} \pi\delta^+(k_1^2)\pi\delta^+(k_2^2)\mathcal{A}_0(k_1, 1, 2, -k_2)\,\mathcal{A}_0(k_2, 3, 4, -k_1)$$

where we notice that a 1-loop amplitude (imaginary part of) has been reduced to the product of tree-level amplitudes, modulus a phase space integration.

A few mores steps . . .

The cutting procedure can be generalized, setting more propagators on-shell to further reduce the $\mathcal{A}_0$ components to fundamental 3- and 4-point tree-level amplitudes.

- 3- and 4-point tree level amplitudes become building blocks: loop contributions obtained by gluing together tree level amplitudes;
- crucial to simplify them as much as possible and use their symmetry properties: helicity/spin state formalism most convenient;
- 3-point on-shell amplitudes vanish: need complex kinematic;
- on-shell internal states avoid redundancy of non-physical states: smaller intermediate expressions;
- most numerical instabilities avoided by automatically combining contributions from several diagrams;
- residual spurious singularity treated in conjunction with the calculation of rational terms (R_1);
- $\mathcal{A}_0$ and $\mathcal{A}_1^{virt}$ calculated as independent complex numbers in a given spinor representation: never to deal with large analytic expressions arising from their interference.

Generalized unitarity leads us to calculate:

$$\mathrm{Im}\,\mathcal{A}_1 = \sum_i d_i\,\mathrm{Im}\,D0_i + \sum_i c_i\,\mathrm{Im}\,C0_i + \sum_i b_i\,\mathrm{Im}\,B0_i + \sum_i a_i\,\mathrm{Im}\,A0_i$$

from which, knowing the analytical expression of the 1-loop integrals, we can determine the coefficients $d_i, \ldots a_i$:

- sets of 4 cuts determine completely each box coefficient d_i;
- sets of 3 cuts determine completely each vertex coefficient c_i, once the 3 cut contributions of the box integrals have been subtracted;
- ...

$$\begin{aligned}
\mathcal{A}_1|_{4-cut_i} &= d_i\, D0_i|_{4-cut} \\
\mathcal{A}_1|_{3-cut_i} &= c_i\, C0_i|_{3-cut} + \sum_j d_j\, D0_j|_{3-cut_i} \\
&\ldots
\end{aligned}$$

Rational terms (R_1), two options

- Using d-dim generalized unitarity automatically include them in $\mathcal{A}^1_{virt}$, but is computationally more demanding.
- Using 4-dim generalized unitarity determines $\mathcal{A}^1_{virt}$ modulus some additive ambiguities due to ultraviolet divergences in the scalar integrals: rational terms (R_1) need to be added to fix this ambiguity. Existing methods make use of: factorization properties of the physical amplitude (Bern,Dixon,Kosower), combined with on-shell recursion relations (Britto,Cachazo,Feng):

$\longrightarrow$ use analytic continuation of the amplitude in complex plane ($\mathcal{A}_1 \to \mathcal{A}_1(z)$), such that $R_1 \to R_1(z)$;

$\longrightarrow$ obtain $R_1(z=0)$ via contour integration in the complex plane:

$$R_1(0) = \int_\gamma \frac{R_1(z)}{z} = \sum_{\text{poles } z_i} \text{Res}|_{z=z_i} \frac{R_1(z)}{z} + R_\infty$$

where R_∞ is the contribution to the contour integral at infinity;

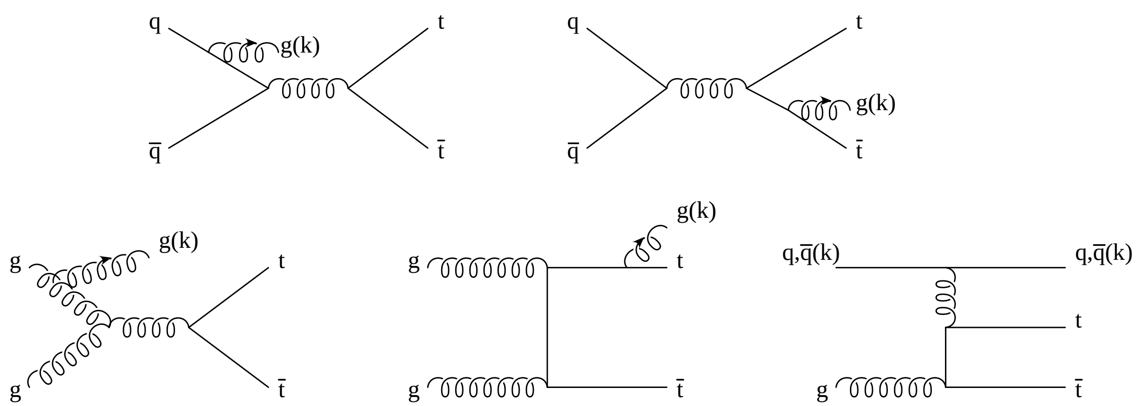
$\longrightarrow$ attention must be paid to distinguish between physical and spurious poles.

$p\bar{p}, pp \to t\bar{t}$: $\mathcal{O}(\alpha_s)$ real corrections

The $\mathcal{O}(\alpha_s)$ real corrections to the cross-section arise from the square of the real gluon/quark emission amplitude $\mathcal{A}_1^{real}(q\bar{q}, gg, qg \to t\bar{t} + (g/q/\bar{q}))$:

$$d\hat{\sigma}_{ij}^{real} = d(PS_{2+(g/q/\bar{q})})\overline{\sum}|\mathcal{A}_1^{real}|^2$$

where $\mathcal{A}_1^{real}$ receives contributions from diagrams like:



IR singularities corresponds to unresolved partons and are extracted isolating the regions of the $t\bar{t} + (g/q/\bar{q})$ phase space where $s_{ik} \to 0$, where:

$$s_{ik} = 2p_i \cdot k = 2E_i k^0 (1 - \beta_i \cos\theta_{ik})$$

- $k^0 \to 0$: soft singularities (both massless and massive particles);
- $\cos\theta_{ik} \to 0$: collinear singularities (massless particles only, $\beta_i = 0$).

Soft and collinear singularities can be isolated thanks to the factorization properties of $|\mathcal{A}_1^{real}|^2$ and $d(PS_{2+(g/q/\bar{q})})$ in the soft and collinear limits.

Consider $\boxed{ij \to t\bar{t} + g}$. In the soft limit ($E_g = k^0 \to 0$):

$$d(PS_{2+g}) \xrightarrow{soft} d(PS_2)d(PS_g) = d(PS_2)\frac{d^{d-1}k}{(2\pi)^{d-1}2k^0}$$

$$\overline{\sum}|\mathcal{A}_1^{real}(ij \to t\bar{t} + g)|^2 \xrightarrow{soft} (4\pi\alpha_s)\Phi_{eik}\overline{\sum}|\mathcal{A}_0|^2$$

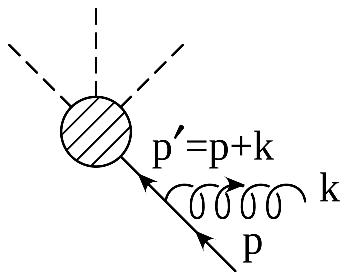
where the eikonal factor Φ_{eik} contains the soft poles ($s_{ij} = 2p_i \cdot p_j$):

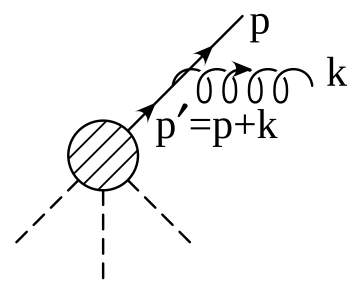
$$\Phi_{eik} \propto \sum_{ij}\left(\frac{s_{ij}}{s_{ik}s_{jk}} - \frac{m_i^2}{s_{ik}^2} - \frac{m_j^2}{s_{jk}^2}\right)$$

$$\Downarrow$$

$$d\hat{\sigma}_{ij}^{soft} \propto d(PS_2)\int_{soft} d(PS_g)\Phi_{eik}\overline{\sum}|\mathcal{A}_0|^2$$

Soft limit, a closer look ...





Calculate $\mathcal{A}_{n+1}$ using:

- incoming line:

$$\cdots \frac{\not{p}' + m}{[(p')^2 - m^2]} \gamma^\mu u(p) \epsilon^*_\mu(k) \xrightarrow{k\to 0} \cdots \frac{\not{p} + m}{2p \cdot k} \gamma^\mu u(p) \epsilon^*_\mu(k) = -i g_s T^a \frac{p \cdot \epsilon^*(k)}{p \cdot k} \mathcal{A}_n$$

- outgoing line:

$$\bar{u}(p) \gamma^\mu \frac{\not{p}' + m}{[(p')^2 - m^2]} \epsilon^*_\mu(k) \ldots \xrightarrow{k\to 0} \bar{u}(p) \gamma^\mu \frac{\not{p} + m}{2p \cdot k} \epsilon^*_\mu(k) \ldots = -i g_s T^a \frac{p \cdot \epsilon^*(k)}{p \cdot k} \mathcal{A}_n$$

When squaring A_{n+1} the eikonal factor appears:

$$\overline{\sum} |\mathcal{A}_{n+1}|^2 = g_s^2 C_F \left| \sum_i \frac{p_i \cdot \epsilon^*(k)}{p_i \cdot k} \right|^2 \overline{\sum} |\mathcal{A}_n|^2 = (4\pi\alpha_s) \Phi_{eik} \overline{\sum} |\mathcal{A}_n|^2$$

In the collinear limit $(i \to i'g,\ p'_i = zp_i,\ k = (1-z)p_i)$

$$\overline{\sum}|\mathcal{A}_1^{real}(ij \to t\bar{t} + g)|^2 \xrightarrow{collinear} (4\pi\alpha_s)\overline{\sum}|\mathcal{A}_0(i'j \to t\bar{t})|^2 \frac{2P_{ii'}(z)}{z\, s_{ik}}$$

$$d(PS_{2+g})(ij \to t\bar{t}) \xrightarrow{collinear} d(PS_2)(i'j \to t\bar{t}) z d(PS_g)$$

$(P_{ii'} \to$ Altarelli-Parisi splitting functions)

$$\Downarrow$$

$$d\hat{\sigma}_{ij}^{hard/coll} \propto d(PS_2) \int_{coll} d(PS_g) \sum_i \frac{P_{ii'}}{s_{ik}} \overline{\sum}|\mathcal{A}_0(i'j \to t\bar{t})|^2$$

The idea is now to calculate analytically only the singular parts of $\hat{\sigma}_{ij}^{real}$, while integrating numerically over the regions of the final state phase space that do not contain singularities. Even more so when we think of calculating processes with several particles (some of which massive) in the final state.

Collinear limit, a closer look . . .

Use collinear kinematics, imposing $k^2 = 0$ (i.e. $k^2 = \mathcal{O}(p_\perp^4)$):

$$
\begin{aligned}
p' &= \left(zp, -p_\perp, 0, zp + \frac{p_\perp^2}{2(1-z)p} \right) \\
k &= \left((1-z)p, p_\perp, 0, (1-z)p - \frac{p_\perp^2}{2(1-z)p} \right)
\end{aligned}
$$

and calculate $|\mathcal{A}_{n+1}^{real}|^2$ using that $(p')^2 = p_\perp^2/(1-z)$, to obtain:

$$
|\mathcal{A}_{n+1}^{real}|^2 \stackrel{collinear}{\longrightarrow} (4\pi\alpha_s) \frac{2P_{qq}(z)}{z\, s_{pk}} |\mathcal{A}_n|^2
$$

where $s_{pk} = 2p \cdot k$ and P_{qq} is $q \to q$ Altarelli-Parisi splitting function.

IR singularities are extracted:

- by imposing suitable cuts on the phase space of the radiated parton: phase space slicing (PSS) method.
  - → PSS with two cutoffs:

    B. W. Harris, J. F. Owens, PRD 65 (2002) 094032 (review paper);
  - → PSS with one cutoff:

    W. T. Giele, E. W. N. Glover, D. A. Kosower, PRD 46 (1992) 1980;

    NPB 403 (1993) 633; S. Keller and E. Laenen, PRD 59 (1999) 114004.
- by using a subtraction method:
  - S. Catani, M. H. Seymour, PLB 378 (1996) 287, NPB 485 (1997) 291 (dipole subtraction);
  - D. A. Kosower PRD 57 (1998) 5410, PRD 67 (2003) 116003, PRD 71 (2005) 045016 (antenna subtraction).

Remaining initial-state IR singularities are absorbed in the PDF's (mass factorization).

PSS: Two Cutoff Method (δ_s,δ_c):

$$d\hat{\sigma}_{ij}^{real}(ij \to t\bar{t} + g) = d\hat{\sigma}_{ij}^{soft} + d\hat{\sigma}_{ij}^{hard/coll} + d\hat{\sigma}_{ij}^{hard/non-coll}$$

where

- $d\hat{\sigma}_{ij}^{soft} \longrightarrow E_g < \frac{\sqrt{s}}{2}\delta_s$
- $d\hat{\sigma}_{ij}^{hard/coll} \longrightarrow E_g > \frac{\sqrt{s}}{2}\delta_s$ and $(1 - \cos\theta_{ik}) < \delta_c$

are computed analytically to extract the IR singularities, while:

- $d\hat{\sigma}_{ij}^{hard/non-coll} \longrightarrow E_g > \frac{\sqrt{s}}{2}\delta_s$ and $(1 - \cos\theta_{ik}) > \delta_c$

can be computed numerically, since is IR finite.

$$\Downarrow$$

The dependence on the cutoffs needs to cancel in the physical cross section.

PSS: One Cutoff Method (s_{min}):

$$d\hat{\sigma}_{ij}^{real}(ij \to t\bar{t} + g) = d\hat{\sigma}_{ij}^{ir} + d\hat{\sigma}_{ij}^{hard}$$

where

- $d\hat{\sigma}_{ij}^{ir} \longrightarrow s_{ik} < s_{min}$

is computed analytically to extract the IR singularities:
$\longrightarrow$ cross all colored particles to final state;
$\longrightarrow$ work with color ordered amplitudes: easier matching between soft and collinear region;
$\longrightarrow$ introduce crossing functions: to account for difference between initial state and final state collinear singularities.

- $d\hat{\sigma}_{ij}^{hard} \longrightarrow s_{ig} > s_{min}$

can be computed numerically, since IR finite.

$$\Downarrow$$

The dependence on the cutoff needs to cancel in the physical cross section.

Consider e.g. $ij \to t\bar{t} + g$:

$$d\hat{\sigma}_{ij}^{real} = d(PS_4)d(PS_g)\overline{\sum}|\mathcal{A}_{ijt\bar{t}+g}^{real}|^2 \ ,$$

where

$$\mathcal{A}_{ijt\bar{t}+g}^{real} = \sum_{\substack{a,b,c \\ i\neq j\neq k}} \mathcal{A}_{abc}\, T^a\, T^b\, T^c \ .$$

$T^a \to$ color matrices, $\mathcal{A}_{abc} \to$ color ordered amplitudes.

The amplitude square is made of three terms:

$$\overline{\sum}|\mathcal{A}_{ijt\bar{t}+g}^{real}|^2 = \overline{\sum}\frac{(N^2-1)}{2}\left[\frac{N^2}{4}\sum_{\substack{a,b,c \\ a\neq b\neq c}}|\mathcal{A}_{abc}|^2\right.$$
$$\left.-\frac{1}{4}\sum_{\substack{a,b,c \\ a\neq b\neq c}}|\mathcal{A}_{abc}+\mathcal{A}_{acb}+\mathcal{A}_{cab}|^2 + \frac{1}{4}\left(1+\frac{1}{N^2}\right)\left|\sum_{\substack{a,b,c \\ a\neq b\neq c}}\mathcal{A}_{abc}\right|^2\right]$$

with very definite soft/collinear factorization properties.

Soft singularities $\longrightarrow$ straightforward

How to disentangle soft vs collinear region of PS with one cutoff?

Collinear limit for $ig \to i'$: $s_{ig} \to 0$ $(i{=}g_1, g_2)$ $\left\{ \begin{array}{l} p_i = z p_i' \\ p_g = (1-z) p_i' \end{array} \right.$

Each $\mathcal{A}_{abc}$ (or linear combination of) proportional to $(s_{ki} s_{ig} s_{gj})^{-1}$

$$\text{collinear region} \left\{ \begin{array}{l} s_{ig} < s_{min} \\ s_{ki} > s_{min} \longrightarrow z s_{ki'} > s_{min} \longrightarrow z > z_1 = \frac{s_{min}}{s_{ki'}} \\ s_{gj} > s_{min} \longrightarrow (1-z) s_{i'j} > s_{min} \longrightarrow z < 1 - z_2 = 1 - \frac{s_{min}}{s_{i'j}} \end{array} \right.$$

$z_1,\ 1 - z_2 \ \longrightarrow$ integration boundaries

How to match with $d\hat{\sigma}_{hard}$? Match each term in $|\mathcal{A}^{real}_{ijt\bar{t}+g}|^2$ separately.

Subtraction Method

Subtract the singular behavior without introducing cutoffs. Schematically:

$$d\hat{\sigma}_{ij}^{NLO} = \left[d\hat{\sigma}_{ij}^{real} - d\hat{\sigma}_{ij}^{sub}\right]_{\epsilon\to 0} + \left[d\hat{\sigma}_{ij}^{virt} + d\hat{\sigma}_{ij}^{sub,CT}\right]_{\epsilon\to 0}$$

where

- $d\hat{\sigma}_{ij}^{sub}$ has the same singular behavior as $d\hat{\sigma}_{ij}^{real}$ at each phase space point (in d dimensions);
- $d\hat{\sigma}_{ij}^{sub}$ has to be analytically integrable over the singular one-parton phase space in d dimensions, such that we can define the subtraction "counterterm":

$$d\hat{\sigma}_{ij}^{sub,CT} = \int d(PS_g)\, d\hat{\sigma}_{ij}^{sub}$$

In this way:

- $[d\hat{\sigma}_{ij}^{real} - d\hat{\sigma}_{ij}^{sub}]$ is integrable over the entire phase space, and the limit $\epsilon \to 0$ can safely be taken;
- $[d\hat{\sigma}_{ij}^{virt} + d\hat{\sigma}_{ij}^{sub,CT}]$ is finite and integrable in $d = 4$ because (modulus the IR singularities that are factored in the renormalized PDF's) $d\hat{\sigma}_{ij}^{sub,CT}$ contains all the IR poles of $d\hat{\sigma}_{ij}^{virt}$.

Example: $d\hat{\sigma}_{ij}^{sub}$ can be built using the so called dipole formalism.

(S. Catani and M.H. Seymour, NPB 485 (1997) 291)

"Dipoles" $\longrightarrow$ soft or collinear singular structures (dV_{dipole}) defined by the factorization of the real emission amplitude ($\rightarrow$ see this lecture).

$$d\hat{\sigma}_{ij}^{sub} = \sum_{dipoles} d\hat{\sigma}_{ij}^{LO} \otimes dV_{dipole}$$

$d\hat{\sigma}_{ij}^{LO} \longrightarrow$ tree level cross-section.

such that:

$$d\hat{\sigma}_{ij}^{sub,CT} = \int d(PS_g)\, d\hat{\sigma}_{ij}^{sub} = d\hat{\sigma}_{ij}^{LO} \otimes \sum_{dipoles} \int d(PS_g)\, dV_{dipole}$$